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Open string states and D-brane tension from vacuum string field theory

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Abstract: We propose a description of open string fields on a D25-brane in vacuum string field theory. We show that the tachyon mass is correctly reproduced from our proposal and further argue that the mass spectrum of all other open string states is correctly obtained as well. We identify the string coupling constant from the three-tachyon coupling and show that the tension of a D25-brane is correctly expressed in terms of the coupling constant, which resolves the controversy in the literature. We also discuss a reformulation of our description which is rather similar to boundary string field theory.

Keywords: Bosonic Strings, Tachyon Condensation, String Field Theory.

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Contents

1. Introduction

One of the most crucial open problems regarding vacuum string field theory (VSFT) [1] is that while ratios of tensions of various D-branes can be reproduced [2], the tension of a single D25-brane has not yet been obtained correctly. Another important problem is that we do not completely understand how to describe open string states around D-branes in the framework of VSFT. The goal of this paper is to provide a resolution to these two problems.

Actually, these two problems are closely related. Since the D25-brane tension T_{25} is related to the on-shell three-tachyon coupling g_T through the relation [3, 4][1]

$$T_{25} = \frac{1}{2\pi^2 \alpha'^3 g_T^2}, \tag{1.1}$$

the energy density $\mathcal{E}_c$ of the classical solution corresponding to a single D25-brane must satisfy

$$\frac{\mathcal{E}_c}{T_{25}} = 2\pi^2 \alpha'^3 g_T^2 \, \mathcal{E}_c = 1\,. \tag{1.2}$$

However, the on-shell three-tachyon coupling g_T based on the earlier proposal for the tachyon state [6] failed to reproduce the relation (1.1) [6, 7], and the ratio $\mathcal{E}_c/T_{25}$ turned out to be [8, 9]

$$\frac{\mathcal{E}_c}{T_{25}} = \frac{\pi^2}{3}\left(\frac{16}{27\ln 2}\right)^3 \simeq 2.0558\,. \tag{1.3}$$

This is regarded as the most crucial problem with the earlier proposal for the tachyon state [6]. If we assume the universality of the ghost part of solutions in VSFT [10], the calculation of the ratio $\mathcal{E}_c/T_{25}$ involves only the matter sector of VSFT. The matter part of the classical solution representing a D25-brane is assumed to be described by the sliver state [10], which is one of the basic assumptions in VSFT. It is generally believed that the wrong ratio (1.3) is not due to the identification of the sliver state as a D25-brane, but originated in the incorrect identification of the tachyon state.

Is there any canonical way to identify open string states around a D-brane solution in VSFT? Let us recall the situation in ordinary field theories. Given a classical solution breaking translational invariance, there must be a massless Goldstone mode around the solution. This massless mode can be identified by infinitesimal deformation of the collective coordinate. If we apply the same logic to a lower-dimensional D-brane solution in VSFT, the massless scalar field on the D-brane should be identified as the associated Goldstone mode. Since a lower-dimensional D-brane is described by a sliver state with a Dirichlet boundary condition [10, 2, 11], infinitesimal deformation of the collective coordinate of the D-brane corresponds to infinitesimal deformation of the Dirichlet boundary condition. As is familiar in the open string sigma model, this is realized by an insertion of the vertex operator of the massless scalar integrated over the boundary of the sliver state.

This is easily generalized to other open string states. In general, an open string state on a D-brane is described by a sliver state where the corresponding vertex operator of the open string state is integrated along the boundary with the boundary condition of the D-brane. Actually, this idea has already been discussed to some extent in [2], but little progress has been made thereafter. We emphasize, however, that it is crucial for this identification to work for the consistency of a series of assumptions regarding VSFT. We will therefore revisit and explore this idea, and show that the open string mass spectrum and the D-brane tension are in fact correctly reproduced.

[1] Since this relation plays an important role in this paper, we verify it using the notation of [5] in appendix A. We use the convention that $\alpha' = 1$ except in the introduction and appendix A.

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Interestingly, the resulting description of open string states turns out to be rather close to that of boundary string field theory (BSFT) or background-independent open string field theory [12]–[16]. In fact, our results are effectively reproduced by a BSFT-like action. We discuss some aspects of this reformulation in this paper.

The organization of this paper is as follows. In section 2 we propose a description of open string fields on a D25-brane in VSFT. We then consider linearized equations of motion for the open string states in section 3 and explain how physical state conditions such as the on-shell condition are imposed. In section 4 we express the string field theory action in terms of the open string fields on a D25-brane. We show that the kinetic terms of the open string fields vanish when the fields satisfy the physical state conditions despite some subtleties. We then calculate the normalization of the tachyon kinetic term and the on-shell three-tachyon interaction to identify the coupling constant g_T, and show that it is correctly related to the D25-brane tension T_{25} through (1.1). Section 5 is devoted to discussions of some problematic issues in our formulation, a BSFT-like reformulation of our description, and future directions.

2. Open string states in vacuum string field theory

The action of VSFT is given by [1]

$$S = -\frac{1}{2}\langle\Psi|\mathcal{Q}|\Psi\rangle - \frac{1}{3}\langle\Psi|\Psi * \Psi\rangle\,, \tag{2.1}$$

where $|\Psi\rangle$ is the string field represented by a state with ghost number one in the boundary conformal field theory (BCFT), and where the definitions of the BPZ inner product $\langle A|B\rangle$ and the star product $|\Psi * \Psi\rangle$ are standard ones [17]. It is conjectured that the state $|\Psi\rangle = 0$ corresponds to the tachyon vacuum, and the operator $\mathcal{Q}$ with ghost number one is made purely of ghost fields. We defined $|\Psi\rangle$ and $\mathcal{Q}$ to absorb an overall normalization factor including the open string coupling constant. We take the matter part of the BCFT to be the one describing a D25-brane.

Classical solutions corresponding to various D-branes are assumed to take the factorized form,

$$|\Psi\rangle = |\Psi_g\rangle \otimes |\Psi_m\rangle\,, \tag{2.2}$$

so that the equation of motion factorizes into

$$\mathcal{Q}\,|\Psi_g\rangle + |\Psi_g * \Psi_g\rangle = 0\,, \tag{2.3}$$

and

$$|\Psi_m\rangle = |\Psi_m * \Psi_m\rangle\,. \tag{2.4}$$

It is further assumed that all D-brane solutions have the same ghost part. On the other hand, the matter part is given by the sliver state with the boundary condition corresponding to the D-brane [2].

For example, a D25-brane is described by the matter part of the sliver state $|\Xi_m\rangle$ with the Neumann boundary condition defined by a limit of the matter part of wedge states $|n\rangle$ [18],

$$|\Xi_m\rangle = \lim_{n\to\infty} |n\rangle \ . \tag{2.5}$$

The matter part of the wedge states $|n\rangle$ is defined by

$$\langle n|\phi\rangle = \mathcal{N} \, \langle f_n \circ \phi(0)\rangle_{\mathrm{UHP}} \ , \tag{2.6}$$

for any state in the matter Fock space $|\phi\rangle$. In (2.6), the conformal transformation $f_n(\xi)$ is given by

$$f_n(\xi) = \frac{n}{2} \tan\left(\frac{2}{n} \tan^{-1} \xi\right) , \tag{2.7}$$

the correlation function is evaluated on the upper-half plane, and $\mathcal{N}$ is an appropriate normalization factor. Here the combination of the conformal transformation $f_n(\xi)$ and the upper-half plane can be replaced by any combination of a conformal transformation $h_n(\xi)$ and Riemann surface Σ_n as long as the combination is conformally equivalent to that of $f_n(\xi)$ and the upper-half plane.

For lower-dimensional Dp-branes, the Neumann sliver state is replaced by the sliver state with the Dirichlet boundary condition for each of the $25-p$ transverse directions. It is similarly defined as in (2.6) by a limit of wedge states with the following boundary condition for the string coordinate $X^i(z)$ on the real axis t of the upper-half plane:[2]

$$\begin{aligned} \partial X^i(t) &= \bar\partial X^i(t) \qquad \text{for } -\frac{n}{2}\tan\frac{\pi}{2n} \le t \le \frac{n}{2}\tan\frac{\pi}{2n} \, , \\ X^i(t) &= a^i \qquad \text{for } t < -\frac{n}{2}\tan\frac{\pi}{2n} \, , \quad t > \frac{n}{2}\tan\frac{\pi}{2n} \, , \end{aligned} \tag{2.8}$$

where a^i is the position of the D-brane in space-time. It is shown in [2] that ratios of tensions of various D-branes are correctly reproduced based on this description.

As we mentioned in the introduction, it is important to identify open string states around D-brane solutions correctly. A proposal for the tachyon state around a D25-brane in VSFT was put forward in the operator formulation [19]–[23] by Hata and Kawano [6]. Its conformal field theory (CFT) description [24, 25] was found to be the sliver state with the tachyon vertex operator[3] e^{ikX} inserted at the midpoint of the boundary of the sliver state [8], which corresponds to $t=\infty$ in our notation. It turned out, however, that the D25-brane tension is not correctly reproduced with this proposal [6, 7, 8, 9]. Since it seems difficult to find the correct description of the tachyon by trial and error, let us take the approach suggested in the introduction.[4]

The massless scalar fields on a D-brane describing its fluctuation in the transverse directions are Goldstone modes associated with the broken translational symmetries, and

[2]For a more complete description, an appropriate regularization is necessary. See [2] for details.

[3]The normal ordering for vertex operators is implicit throughout the paper.

[4]For other approaches along the line of the CFT description [8] of the Hata-Kawano tachyon state [6], see [26, 27, 28]. Possibilities of identifying D-branes as classical solutions other than sliver-type configurations were studied in [29, 30].

should be identified with infinitesimal deformations of the collective coordinates. Since the collective coordinates are encoded as Dirichlet boundary conditions $X^i = a^i$ in (2.8), the massless scalar fields must be identified with infinitesimal deformations of the boundary condition. It is well-known in the open string sigma model that such a deformation is realized by an insertion of an integral of the vertex operator $\partial_\perp X^i e^{ikX}$ of the scalar field, where $\partial_\perp$ is the derivative normal to the boundary. Therefore, the massless scalar field should be described by the sliver state where the integral of the vertex operator is inserted along the boundary with the Dirichlet boundary condition.

This identification of the scalar fields is generalized to other open string states using the relation between the deformation of the boundary condition and the insertion of an integrated vertex operator. For example, the tachyon field $T(k)$ on a D25-brane represented by the sliver state $|\Xi_m\rangle$ (2.5) should be described at the linear order of $T(k)$ as follows:

$$|\Psi_m\rangle = |\Xi_m\rangle - \int d^{26}k \; T(k)\, |\chi_T(k)\rangle \;, \tag{2.9}$$

where $|\chi_T(k)\rangle$ is defined for any state in the matter Fock space $|\phi\rangle$ by

$$\langle \chi_T(k)|\phi\rangle = \lim_{n\to\infty} \mathcal{N} \left\langle \int dt\, e^{ikX(t)} h_n \circ \phi(0) \right\rangle_{\Sigma_n} . \tag{2.10}$$

Here a combination of a conformal transformation $h_n(\xi)$ and Riemann surface Σ_n should be conformally equivalent to that of $f_n(\xi)$ and the upper-half plane, and the integral of the tachyon vertex operator $e^{ikX(t)}$ is taken along the boundary of the wedge state from $h_n(1)$ to $h_n(-1)$.[5] Since the integrated vertex operator is not generically conformally invariant, the definition of the tachyon state depends on the choice of $(h_n(\xi),\, \Sigma_n)$. Note, however, that this ambiguity is absent if the tachyon is on shell, $k^2 = 1$, because the integral of the vertex operator becomes conformally invariant. Therefore, the ambiguity coming from the choice of $(h_n(\xi),\, \Sigma_n)$ can be regarded as that of field redefinition of the tachyon field.

This ambiguity can also be described as follows. We can make a conformal transformation such that the Riemann surface Σ_n, where the off-shell tachyon field is defined in (2.10), is mapped to a cone subtending an angle $n\pi$ at the origin. Its boundary is parametrized as $e^{i\theta}$ with $-n\pi/2 \le \theta \le n\pi/2$, and we choose the region of the integral of the vertex operator to be $-(n-1)\pi/2 \le \theta \le (n-1)\pi/2$.[6] The inserted operator now takes the form

$$\int_{-(n-1)\pi/2}^{(n-1)\pi/2} d\theta \mathcal{F}_0(\theta)^{k^2-1} e^{ikX(e^{i\theta})} \,, \tag{2.11}$$

where the additional factor $\mathcal{F}_0(\theta)^{k^2-1}$ comes from the conformal transformation from Σ_n

[5] In the case of the upper-half plane, for example, the integral is taken along

$$\int_{f_n(1)}^{\infty} dt + \int_{-\infty}^{f_n(-1)} dt = \int_{\frac{n}{2}\tan\frac{\pi}{2n}}^{\infty} dt + \int_{-\infty}^{-\frac{n}{2}\tan\frac{\pi}{2n}} dt.$$

[6] In other words, the boundary of the local coordinate ξ [2], which is $-1 \le \xi \le 1$ on the real axis, is mapped to the sum of the two regions $(n-1)\pi/2 \le \theta \le n\pi/2$ and $-n\pi/2 \le \theta \le -(n-1)\pi/2$.

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to the cone. The ambiguity of the off-shell definition of the tachyon is now encoded in this factor. We will frequently use this representation of off-shell tachyon configurations in the rest of the paper and refer to this as *the cone representation*.

So far we have considered infinitesimal deformations of the BCFT. It is easily generalized to finite deformations as follows:

$$\langle \{\varphi_i\} | \phi \rangle = \lim_{n\to\infty} \mathcal{N} \left\langle \exp\left[-\int dt \int d^{26}k \sum_i \varphi_i(k) \mathcal{O}_{\varphi_i(k)}(t) \right] h_n \circ \phi(0) \right\rangle_{\Sigma_n} , \tag{2.12}$$

where $\{\varphi_i\}$ denotes the open string fields on a D25-brane such as the tachyon $T(k)$ or the massless gauge field $A_\mu(k)$ collectively, $\mathcal{O}_{\varphi_i(k)}$ is the vertex operator corresponding to the field $\varphi_i(k)$, and the integral over t is taken along the boundary as before. This is a formal definition because we need to regularize divergences which appear when some of the operators $\mathcal{O}_{\varphi_i(k)}$ coincide. We will come back to this point later in sections 4 and 5. There would also be some ambiguity when we assign vertex operators $\mathcal{O}_{\varphi_i(k)}$ to fields $\varphi_i(k)$ when they are off shell. It would be natural, however, to associate e^{ikX} with the tachyon because it is primary even when the momentum k is off shell.

3. Equations of motion for open string states

Let us consider the linearized equations of motion for the open string fields $\{\varphi_i\}$ to see if our identification of the open string states on a single D25-brane works:

$$|\chi_{\varphi_i}(k)\rangle = |\chi_{\varphi_i}(k) * \Xi_m\rangle + |\Xi_m * \chi_{\varphi_i}(k)\rangle , \tag{3.1}$$

where

$$\langle \chi_{\varphi_i}(k) | \phi \rangle = \lim_{n\to\infty} \mathcal{N} \left\langle \int dt\, \mathcal{O}_{\varphi_i}(t)\, h_n \circ \phi(0) \right\rangle_{\Sigma_n} , \tag{3.2}$$

for any state in the matter Fock space $|\phi\rangle$ and the integral over t is the same as (2.10).

As we will see in more detail shortly, the sum of the two integrals of the vertex operator on the right-hand side of (3.1) gives the integral on the left-hand side in the large-n limit. Therefore, it is easily understood that the equation of motion (3.1) can be satisfied. What is less obvious is how the physical state conditions are imposed. For example, the on-shell condition $k^2 = 1$ must be imposed for the tachyon, and the massless condition $k^2 = 0$ and the transversality condition must be imposed for the gauge field. Let us take a closer look at the large-n limit by considering the following inner products $\mathcal{A}_L$ and $\mathcal{A}_R$ defined by

$$\mathcal{A}_L = \langle \phi | \chi_{\varphi_i}(k) \rangle_n , \qquad \mathcal{A}_R = \langle \phi | \left(|\chi_{\varphi_i}(k)\rangle_n * |n\rangle + |n\rangle * |\chi_{\varphi_i}(k)\rangle_n \right) , \tag{3.3}$$

where $|\phi\rangle$ is an arbitrary state in the matter Fock space and the large-n limit is not taken for the open string state $|\chi_{\varphi_i}(k)\rangle_n$ as is indicated by the subscript n. The equation of motion (3.1) contracted with $\langle \phi|$ is given by

$$\lim_{n\to\infty} \mathcal{A}_L = \lim_{n\to\infty} \mathcal{A}_R . \tag{3.4}$$

Let us first consider the case without the integral of the vertex operator along the boundary. The inner products $\mathcal{A}_L$ and $\mathcal{A}_R$ then reduce to

$$\mathcal{A}_L \to \langle\phi|n\rangle\ , \qquad \mathcal{A}_R \to 2\,\langle\phi|n*n\rangle = 2\,\langle\phi|2n-1\rangle\ , \tag{3.5}$$

where we used the famous star algebra of wedge states [18]:

$$|n\rangle * |m\rangle = |n+m-1\rangle\,. \tag{3.6}$$

What is important here is that the two conformal transformations $f_n(\xi)$ and $f_{2n-1}(\xi)$ used in defining $|n\rangle$ and $|2n-1\rangle$, respectively, have the same large-n limit:

$$\lim_{n\to\infty} f_n(\xi) = \lim_{n\to\infty} f_{2n-1}(\xi) = \tan^{-1}\xi\,. \tag{3.7}$$

This was essential for the sliver state to solve the matter equation of motion of VSFT [2].

Now consider the effect of the integrated vertex operator. Let us take the case of the tachyon as an example. The inner product $\mathcal{A}_L$ is given by

$$\mathcal{A}_L = \langle\phi|\chi_T(k)\rangle_n = \mathcal{N}\left\langle \int_C dt\, e^{ikX(t)}\mathcal{F}_L(t)^{k^2-1}\, f_n\circ\phi(0)\right\rangle_{\rm UHP}\,, \tag{3.8}$$

where the contour C is given by

$$\int_C dt = \int_{f_n(1)}^{\infty} dt + \int_{-\infty}^{f_n(-1)} dt = \int_{\frac{n}{2}\tan\frac{\pi}{2n}}^{\infty} dt + \int_{-\infty}^{-\frac{n}{2}\tan\frac{\pi}{2n}} dt\,. \tag{3.9}$$

Note that we inserted a factor $\mathcal{F}_L(t)^{k^2-1}$ which comes from a conformal transformation from the Riemann surface Σ_n, where the off-shell tachyon is defined, to the upper-half plane. In constructing the states $|\chi_T(k)\rangle_n * |n\rangle$ and $|n\rangle * |\chi_T(k)\rangle_n$, the vertex operator undergoes a further conformal transformation. The contour C (3.9) is mapped to

$$\int_{C_R} dt = \int_{f_{2n-1}(1)}^{\infty} dt = \int_{\frac{2n-1}{2}\tan\frac{\pi}{2(2n-1)}}^{\infty} dt \tag{3.10}$$

for $|\chi_T(k)\rangle_n * |n\rangle$, and to

$$\int_{C_L} dt = \int_{-\infty}^{f_{2n-1}(-1)} dt = \int_{-\infty}^{-\frac{2n-1}{2}\tan\frac{\pi}{2(2n-1)}} dt \tag{3.11}$$

for $|n\rangle * |\chi_T(k)\rangle_n$. The inner product $\mathcal{A}_R$ is then given by

$$\begin{aligned}\mathcal{A}_R &= \langle\phi|\left(|\chi_T(k)\rangle_n * |n\rangle + |n\rangle * |\chi_T(k)\rangle_n\right)\\ &= \mathcal{N}\left\langle \int_{C'} dt\, e^{ikX(t)}\mathcal{F}_R(t)^{k^2-1} f_{2n-1}\circ\phi(0)\right\rangle_{\rm UHP}\,,\end{aligned} \tag{3.12}$$

where the contour C' is the sum of C_R (3.10) and C_L (3.11). Note that the factor $\mathcal{F}_R(t)^{k^2-1}$ in $\mathcal{A}_R$ is different from the one $\mathcal{F}_L(t)^{k^2-1}$ in $\mathcal{A}_L$ because of the additional conformal transformations. Therefore, although $f_n(\xi)$ and $f_{2n-1}(\xi)$ have the same large-n limit and the contours C and C' become the same in the large-n limit, the large-n limit of $\mathcal{A}_L$ and that of $\mathcal{A}_R$ do not coincide because of the difference between $\mathcal{F}_L(t)^{k^2-1}$ and $\mathcal{F}_R(t)^{k^2-1}$ *unless* the condition $k^2 = 1$ is satisfied. This shows that the tachyon state $|\chi_T(k)\rangle$ satisfies the equation of motion only when $k^2 = 1$, which is the correct on-shell condition for the tachyon.

The origin of the condition $k^2 = 1$ is obvious: it is the condition that the integral of the vertex operator e^{ikX} is conformally invariant. In other words, the vertex operator must be a primary field with conformal dimension one, which is nothing but the physical state condition for the vertex operator in string theory. The same argument applies to all other open string states: the equation of motion is satisfied when the vertex operator $\mathcal{O}_{\varphi_i(k)}$ is a primary field with conformal dimension one. For the gauge field, this imposes the transversality condition as well as the on-shell condition. The vertex operator for the gauge field $\zeta_\mu \partial_t X^\mu e^{ikX}$ does not transform as a tensor unless the transversality condition $\zeta \cdot k = 0$ is satisfied. The non-tensor property would result in different expressions for $\mathcal{A}_L$ and $\mathcal{A}_R$ which violate the equation of motion as in the case of an off-shell momentum.

4. String field theory action in terms of open string fields

Based on our proposal for the description of the open string fields $|\{\varphi_i\}\rangle$ (2.12), we can rewrite the VSFT action in terms of the open string fields $\{\varphi_i\}$ as follows:

$$S\left[\{\varphi_i\}\right] = -\langle \Psi_g | \mathcal{Q} | \Psi_g \rangle \left[\frac{1}{2} \langle \{\varphi_i\} | \{\varphi_i\} \rangle - \frac{1}{3} \langle \{\varphi_i\} | \{\varphi_i\} * \{\varphi_i\} \rangle \right]. \tag{4.1}$$

The kinetic terms of the open string fields might be expected to be reproduced correctly in view of the argument presented in the previous section. However, it is not automatic. We have shown that the states $|\chi_{\varphi_i}(k)\rangle$ satisfy the equations of motion (3.1) when contracted with any state in the matter Fock space $|\phi\rangle$. The proof can be generalized to cases where the equations of motion (3.1) are contracted with a larger class of states such as wedge states with some operators inserted as long as the size of the wedge stays finite while we take the large-n limit for $|\chi_{\varphi_i}(k)\rangle$ and $|\Xi_m\rangle$. When we evaluate the kinetic terms of the fields $\{\varphi_i\}$ in (4.1), however, we have to handle the following combination of inner products:

$$\langle \chi_{\varphi_i}(-k) | \chi_{\varphi_i}(k) \rangle - \langle \chi_{\varphi_i}(-k) | \chi_{\varphi_i}(k) * \Xi_m \rangle - \langle \chi_{\varphi_i}(-k) | \Xi_m * \chi_{\varphi_i}(k) \rangle . \tag{4.2}$$

This takes the form of the equation of motion (3.1) contracted with $\langle \chi_{\varphi_i}(-k)|$, but we cannot apply the argument in the previous section to this case because of two subtleties. First, the expression contains divergences when two vertex operators coincide. Secondly, we have to take the large-n limit for $\langle \chi_{\varphi_i}(-k)|$ as well so that the state $\langle \chi_{\varphi_i}(-k)|$ does not belong to the class of states we just mentioned. Therefore, it is important to verify whether the correct kinetic terms are reproduced for the consistency of our proposal. In fact, it was argued in [8] that this is where the earlier proposal [6] for the tachyon state failed. In subsection 4.1 we will show that the kinetic terms of the fields $\{\varphi_i\}$ in (4.1) vanish when they satisfy the physical state conditions.

As we mentioned in the introduction, we are particularly interested in the on-shell three-tachyon coupling g_T which is related to the D25-brane tension through (1.1). We will calculate the normalization of the tachyon kinetic term in subsection 4.2 and the on-shell three-tachyon interaction in subsection 4.3 to show that the correct D25-brane tension is actually reproduced from (4.1).

4.1 Open string mass spectrum

Let us begin with the tachyon and set other fields to zero in $|\{\varphi_i\}\rangle$ (2.12) for simplicity. If we denote the resulting state representing tachyon field configurations by $|e^{-T}\rangle$,

$$\langle e^{-T}|\phi\rangle = \lim_{n\to\infty} \mathcal{N} \left\langle \exp\left[-\int dt \int d^{26}k\ T(k) e^{ikX}(t) \right] h_n \circ \phi(0) \right\rangle_{\Sigma_n}, \tag{4.3}$$

the action for the tachyon field is given by

$$S\left[T(k)\right] = -\left\langle \Psi_g|\mathcal{Q}|\Psi_g\right\rangle \left[\frac{1}{2}\langle e^{-T}|e^{-T}\rangle - \frac{1}{3}\langle e^{-T}|e^{-T} * e^{-T}\rangle \right]. \tag{4.4}$$

The state $|e^{-T}\rangle$ has nonlinear dependence on the tachyon field $T(k)$. If we expand it in powers of $T(k)$, we have

$$|e^{-T}\rangle = |\mathit{0}\rangle + |\mathit{1}\rangle + |\mathit{2}\rangle + |\mathit{3}\rangle + \cdots, \tag{4.5}$$

where[7]

$$|\mathit{0}\rangle = |\Xi_m\rangle, \qquad |\mathit{1}\rangle = -\int d^{26}k\ T(k)\, |\chi_T(k)\rangle, \ldots \tag{4.6}$$

The tachyon kinetic term is given by

$$S^{(2)} = -\left\langle \Psi_g|\mathcal{Q}|\Psi_g\right\rangle \left[\frac{1}{2}\langle \mathit{1}|\mathit{1}\rangle + \langle \mathit{2}|\mathit{0}\rangle - \langle \mathit{1}|\mathit{1} * \mathit{0}\rangle - \langle \mathit{2}|\mathit{0} * \mathit{0}\rangle \right]. \tag{4.7}$$

This takes the following form:

$$S^{(2)} = -\frac{\mathcal{K}}{2}(2\pi)^{26} \int d^{26}k\ K(k^2) T(k) T(-k), \tag{4.8}$$

where we denote the density of $\langle \Xi_m|\Xi_m\rangle \langle \Psi_g|\mathcal{Q}|\Psi_g\rangle$ by $\mathcal{K}$, namely,

$$\langle \Xi_m|\Xi_m\rangle \langle \Psi_g|\mathcal{Q}|\Psi_g\rangle = \int d^{26}x \mathcal{K}, \tag{4.9}$$

and $K(k^2)$ consists of contributions from $\langle \mathit{1}|\mathit{1}\rangle$, $\langle \mathit{2}|\mathit{0}\rangle$, $\langle \mathit{1}|\mathit{1} * \mathit{0}\rangle$, and $\langle \mathit{2}|\mathit{0} * \mathit{0}\rangle$,

$$\frac{1}{2}K(k^2) = \frac{1}{2}K_{11}(k^2) + K_{20}(k^2) - K_{110}(k^2) - K_{200}(k^2). \tag{4.10}$$

[7] These states with italic numbers, $|\mathit{0}\rangle, |\mathit{1}\rangle, |\mathit{2}\rangle, \ldots$, should not be confused with wedge states $|0\rangle, |1\rangle, |2\rangle, \ldots$.

It is naively expected that $\langle \mathit{2}|\mathit{0}\rangle - \langle \mathit{2}|\mathit{0} * \mathit{0}\rangle$ vanishes because of the equation of motion for the sliver state, $|\mathit{0} * \mathit{0}\rangle = |\mathit{0}\rangle$, and the remaining combination of terms,

$$\frac{1}{2}\langle \mathit{1}|\mathit{1}\rangle - \langle \mathit{1}|\mathit{1} * \mathit{0}\rangle = \frac{1}{2}\left[\langle \mathit{1}|\mathit{1}\rangle - \langle \mathit{1}|\mathit{1} * \mathit{0}\rangle - \langle \mathit{1}|\mathit{0} * \mathit{1}\rangle\right], \tag{4.11}$$

vanishes when the tachyon is on shell because of the equation of motion for $|\chi_T(k)\rangle$. However, it turns out that the story is more complicated because of the subtleties we mentioned at the beginning of this section.

Let us begin with the calculations of $\langle \mathit{1}|\mathit{1}\rangle$ and $\langle \mathit{1}|\mathit{1} * \mathit{0}\rangle$. To avoid the singularity which arises when the two vertex operators coincide, we regularize the state $|\mathit{1}\rangle$. We use the cone representation of $|\mathit{1}\rangle$ explained in the paragraph containing (2.11) and regularize the integral of the vertex operator as follows:

$$\int_{-(n-1)\pi/2+\epsilon_0/2}^{(n-1)\pi/2-\epsilon_0/2} d\theta\, \mathcal{F}_0(\theta)^{k^2-1} e^{ikX(e^{i\theta})}\,. \tag{4.12}$$

The inner product $\langle \phi|\mathit{1}\rangle$ is expressed on a cone with an angle $n\pi$. To construct the inner product $\langle \mathit{1}|\mathit{1}\rangle$, we cut off the region of the cone where the local coordinate is mapped, which leaves a sector of an angle $(n-1)\pi$. By gluing two such sectors together, $\langle \mathit{1}|\mathit{1}\rangle$ is expressed on a cone with an angle $2(n-1)\pi$. We can map it to the unit disk by the conformal transformation $z^{1/(n-1)}$. Using the propagator on the unit disk,

$$\left\langle e^{ikX(e^{i\theta_1})} e^{ik'X(e^{i\theta_2})} \right\rangle_{\rm disk} = (2\pi)^{26}\delta(k+k') \left| 2\sin\frac{\theta_2-\theta_1}{2}\right|^{2kk'}, \tag{4.13}$$

$K_{11}(k^2)$ is given by

$$\begin{aligned} K_{11}(k^2) = &\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi}^{-\pi/2-\epsilon/2} d\theta_1\, \mathcal{F}(\theta_1+\pi)^{k^2-1}\mathcal{F}(\theta_2)^{k^2-1}\left|2\sin\frac{\theta_2-\theta_1}{2}\right|^{-2k^2} + \\ &+ \int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\pi/2+\epsilon/2}^{\pi} d\theta_1\, \mathcal{F}(\theta_1-\pi)^{k^2-1}\mathcal{F}(\theta_2)^{k^2-1}\left|2\sin\frac{\theta_2-\theta_1}{2}\right|^{-2k^2}, \end{aligned} \tag{4.14}$$

where

$$\epsilon = \frac{\epsilon_0}{n-1}, \qquad \mathcal{F}(\theta) = \frac{1}{n-1}\mathcal{F}_0((n-1)\theta)\,. \tag{4.15}$$

The difference between $\mathcal{F}_0$ and $\mathcal{F}$ comes from the conformal transformation $z^{1/(n-1)}$. Since ϵ goes to zero when we take the large-n limit, we do not need to take the limit $\epsilon_0 \to 0$ as in the case of the similar regularizations discussed in [2].

The construction of $\langle \mathit{1}|\mathit{1} * \mathit{0}\rangle$ can be done in a similar way. By gluing together three sectors with an angle $(n-1)\pi$ coming from two $|\mathit{1}\rangle$'s and one $|\mathit{0}\rangle$, $\langle \mathit{1}|\mathit{1} * \mathit{0}\rangle$ is expressed on a cone with an angle $3(n-1)\pi$. We can make a conformal transformation so that the cone is mapped to the unit disk. However, it is more convenient to make the same conformal transformation $z^{1/(n-1)}$ as the case of $\langle \mathit{1}|\mathit{1}\rangle$, which maps the cone with an angle $3(n-1)\pi$ to a cone with an angle 3π. The propagator on this cone is given by

$$\left\langle e^{ikX(e^{i\theta_1})} e^{ik'X(e^{i\theta_2})} \right\rangle_{3\pi} = (2\pi)^{26}\delta(k+k') \left| 3\sin\frac{\theta_2-\theta_1}{3}\right|^{2kk'}, \tag{4.16}$$

which respects the periodicity $\theta_i = \theta_i + 3\pi$, and $K_{110}(k^2)$ is given by

$$K_{110}(k^2) = \int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-3\pi/2+\epsilon/2}^{-\pi/2-\epsilon/2} d\theta_1 \; \mathcal{F}(\theta_1+\pi)^{k^2-1}\mathcal{F}(\theta_2)^{k^2-1} \left| 3\sin\frac{\theta_2-\theta_1}{3} \right|^{-2k^2} . \tag{4.17}$$

Let us calculate $K_{11}(k^2)/2 - K_{110}(k^2)$ when the tachyon is on shell, $k^2 = 1$, to see if it actually vanishes as naively expected. The expressions for $K_{11}(k^2)$ and $K_{110}(k^2)$ simplify when $k^2 = 1$ so that the integrals in $K_{11}(1)$ and $K_{110}(1)$ are easily performed to give

$$\begin{aligned} \frac{1}{2}K_{11}(1) &= \frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi}^{-\pi/2-\epsilon/2} d\theta_1 \; \left| 2\sin\frac{\theta_2-\theta_1}{2} \right|^{-2} + \\ &\quad + \frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\pi/2+\epsilon/2}^{\pi} d\theta_1 \; \left| 2\sin\frac{\theta_2-\theta_1}{2} \right|^{-2} \\ &= -\ln\sin\frac{\epsilon}{2} = -\ln\epsilon + \ln 2 + O(\epsilon^2) , \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} K_{110}(1) &= \int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-3\pi/2+\epsilon/2}^{-\pi/2-\epsilon/2} d\theta_1 \; \left| 3\sin\frac{\theta_2-\theta_1}{3} \right|^{-2} \\ &= -\ln\sin\frac{\epsilon}{3} + 2\ln\sin\frac{\pi}{3} - \ln\sin\left(\frac{\pi}{3}+\frac{\epsilon}{3}\right) \\ &= -\ln\epsilon + \ln\frac{3\sqrt{3}}{2} + O(\epsilon) . \end{aligned} \tag{4.19}$$

Therefore, the divergent part in $K_{11}(1)/2 - K_{110}(1)$ vanishes, but the finite part remains:

$$\frac{1}{2}K_{11}(1) - K_{110}(1) = -\ln\frac{3\sqrt{3}}{4} + O(\epsilon) . \tag{4.20}$$

Does this imply the breakdown of the tachyon equation of motion?

Recall, however, that there are other contributions to the tachyon kinetic term, namely, $K_{20}(k^2)$ and $K_{200}(k^2)$ in (4.10). Let us calculate them. The state $|2\rangle$ needs to be regularized. We regularize the integrals of the inserted vertex operators in the cone representation as follows:

$$\int_{-(n-1)\pi/2+3\epsilon_0/2}^{(n-1)\pi/2-\epsilon_0/2} d\theta_2 \int_{-(n-1)\pi/2+\epsilon_0/2}^{\theta_2-\epsilon_0} d\theta_1 \; \mathcal{F}_0(\theta_1)^{k^2-1} e^{ikX(e^{i\theta_1})} \mathcal{F}_0(\theta_2)^{k^2-1} e^{ikX(e^{i\theta_2})} . \tag{4.21}$$

The inner products $\langle \mathit{2}|\mathit{0}\rangle$ and $\langle \mathit{2}|\mathit{0} * \mathit{0}\rangle$ are constructed similarly as in the cases of $\langle \mathit{1}|\mathit{1}\rangle$ and $\langle \mathit{1}|\mathit{1} * \mathit{0}\rangle$. The resulting expressions for $K_{20}(k^2)$ and $K_{200}(k^2)$ are given by

$$K_{20}(k^2) = \int_{-\pi/2+3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi/2+\epsilon/2}^{\theta_2-\epsilon} d\theta_1 \, \mathcal{F}(\theta_1)^{k^2-1}\mathcal{F}(\theta_2)^{k^2-1} \left| 2\sin\frac{\theta_2-\theta_1}{2} \right|^{-2k^2} , \tag{4.22}$$

and

$$K_{200}(k^2) = \int_{-\pi/2+3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi/2+\epsilon/2}^{\theta_2-\epsilon} d\theta_1 \, \mathcal{F}(\theta_1)^{k^2-1}\mathcal{F}(\theta_2)^{k^2-1} \left| 3\sin\frac{\theta_2-\theta_1}{3} \right|^{-2k^2} . \tag{4.23}$$

Their on-shell values, $K_{20}(1)$ and $K_{200}(1)$, are again easily calculated. Since

$$\begin{aligned}
\int_{-\pi/2+3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi/2+\epsilon/2}^{\theta_2-\epsilon} d\theta_1 & \left| 2n \sin\frac{\theta_2-\theta_1}{2n}\right|^{-2} = \\
&= \frac{\pi-2\epsilon}{2n}\cot\frac{\epsilon}{2n} + \ln\sin\frac{\epsilon}{2n} - \ln\sin\left(\frac{\pi}{2n}-\frac{\epsilon}{2n}\right) \\
&= \frac{\pi}{\epsilon} + \ln\epsilon - \ln\left(2n\sin\frac{\pi}{2n}\right) - 2 + O(\epsilon)\,, \qquad (4.24)
\end{aligned}$$

for $n \geq 1$, we have

$$K_{20}(1) = \frac{\pi}{\epsilon} + \ln\epsilon - \ln 2 - 2 + O(\epsilon)\,, \qquad (4.25)$$

$$K_{200}(1) = \frac{\pi}{\epsilon} + \ln\epsilon - \ln\frac{3\sqrt{3}}{2} - 2 + O(\epsilon)\,. \qquad (4.26)$$

The divergent part in $K_{20}(1) - K_{200}(1)$ vanishes, but the finite part again remains:

$$K_{20}(1) - K_{200}(1) = \ln\frac{3\sqrt{3}}{4} + O(\epsilon)\,. \qquad (4.27)$$

However, this precisely cancels $K_{11}(1)/2 - K_{110}(1)$ in the limit $\epsilon \to 0$. Therefore, the tachyon kinetic term vanishes when $k^2 = 1$ in the large-n limit,

$$\frac{1}{2}K(1) = \frac{1}{2}K_{11}(1) + K_{20}(1) - K_{110}(1) - K_{200}(1) = O(\epsilon)\,, \qquad (4.28)$$

which is the property we expect for the correct tachyon kinetic term.

The cancellation of the finite terms in $K_{11}(1)/2 - K_{110}(1)$ and $K_{20}(1) - K_{200}(1)$ may seem rather accidental and, apparently, it seems to have nothing to do with the argument for the mass spectrum in section 3 based on the conformal invariance of the integrated vertex operator. However, the cancellation can be regarded as a consequence of the conformal property of the vertex operators, as we will show in appendix C using some results from appendix B. Note also that the finite term in (4.27) does not depend on details of the regularization of the state $|2\rangle$. We obtain the same result if we regularize the integrals in (4.22) and (4.23) as

$$\int_{-\pi/2+\epsilon+\eta}^{\pi/2-\eta} d\theta_2 \int_{-\pi/2+\eta}^{\theta_2-\epsilon} d\theta_1 \qquad (4.29)$$

as long as ϵ and η go to zero in the limit.

Next consider open string fields other than the tachyon. The calculation for the tachyon depends only on the two-point functions (4.13) and (4.16), and two-point functions of primary fields are uniquely determined by their conformal dimensions. Therefore, we conclude that the kinetic terms of the open string fields $\{\varphi_i\}$ vanish when the corresponding vertex operator $\mathcal{O}_{\varphi_i(k)}$ is primary and has conformal dimension one. This condition is nothing but the familiar physical state condition in string theory so that the result in this subsection provides strong evidence that the correct mass spectrum of open string states can be obtained in VSFT based on our description of the open string fields $\{\varphi_i\}$.

Note that conditions other than the on-shell condition for a vertex operator to be physical, such as the transversality condition for the massless vector field, are also imposed as we discussed at the end of section 3. The argument so far, however, does not guarantee that the kinetic terms of the open string fields $\{\varphi_i\}$ are correctly reproduced. For example, it is not obvious that the kinetic term for the massless gauge field takes a gauge-invariant form. Any kinetic term of the form

$$A_\mu(k)\left[a(k^2)\eta^{\mu\nu}k^2 + b(k^2)k^\mu k^\nu\right]A_\nu(-k) \tag{4.30}$$

vanishes when $k^2 = 0$ and $k \cdot A(k) = 0$ for any pair of functions $a(k^2)$ and $b(k^2)$. Gauge invariance requires that $b(k^2) = -a(k^2)$. It would be interesting to see if the gauge invariance of the string field theory guarantees the gauge invariance of the open string fields $\{\varphi_i\}$.[8]

There are other issues we have to address regarding the argument in this subsection, which we will discuss in the next section.

4.2 Tachyon kinetic term

In order to read off the on-shell three-tachyon coupling from the cubic interaction, we need to normalize the tachyon field canonically. We have to calculate the coefficient in front of $k^2 - 1$ in $K(k^2)$, but the calculation is much more complicated than that of $K(1)$ because of the conformal factor $\mathcal{F}(\theta)^{k^2-1}$ and the k-dependence in the propagator. We present details of the calculation in appendix B, and explain an outline of the derivation in this subsection.

If we could set $\epsilon = 0$, the integrals of the vertex operators would be over the whole boundary for each term in $S[T(k)]$ (4.4). In that case, $K(k^2)$ is given by

$$\begin{aligned} K(k^2)\Big|_{\epsilon=0} = & \frac{1}{2}\int_0^{2\pi} d\theta_2 \int_0^{2\pi} d\theta_1\, \mathcal{F}(\theta_1)^{k^2-1}\mathcal{F}(\theta_2)^{k^2-1}\left|2\sin\frac{\theta_2-\theta_1}{2}\right|^{-2k^2} - \\ & -\frac{1}{3}\int_0^{3\pi} d\theta_2 \int_0^{3\pi} d\theta_1\; \mathcal{F}(\theta_1)^{k^2-1}\mathcal{F}(\theta_2)^{k^2-1}\left|3\sin\frac{\theta_2-\theta_1}{3}\right|^{-2k^2}, \end{aligned} \tag{4.31}$$

where we have extended the definition of $\mathcal{F}(\theta)$ from $-\pi/2 \leq \theta \leq \pi/2$ to all θ through $\mathcal{F}(\theta+\pi) = \mathcal{F}(\theta)$. If we define $K(n, k^2)$ by

$$K(n,k^2) \equiv \frac{1}{2n}\int_0^{2n\pi} d\theta_2 \int_0^{2n\pi} d\theta_1\, \mathcal{F}(\theta_1)^{k^2-1}\mathcal{F}(\theta_2)^{k^2-1}\left|2n\sin\frac{\theta_2-\theta_1}{2n}\right|^{-2k^2}, \tag{4.32}$$

$K(k^2)$ is given by

$$K(k^2)\Big|_{\epsilon=0} = K(1,k^2) - K\left(\frac{3}{2},k^2\right). \tag{4.33}$$

[8]We can confirm, for example, that $|\chi_{A_\mu(k)+k_\mu}\rangle$ and $|\chi_{A_\mu(k)}\rangle$ are gauge equivalent when $k^2 = 0$ and $k \cdot A = 0$ in the framework of section 3, namely, as far as we consider inner products with wedge states which remain finite in the large-n limit. However, it is not clear if it holds in the framework of the present section.

Let us calculate $K(n, k^2)$. Since we are interested in the region $k^2 \simeq 1$, we expand the conformal factors around $k^2 = 1$ to find

$$\begin{aligned}
K(n,k^2) &= \frac{1}{2n}\int_0^{2n\pi} d\theta_2 \int_0^{2n\pi} d\theta_1 \left|2n\sin\frac{\theta_2-\theta_1}{2n}\right|^{-2k^2} \times \\
&\qquad \times \left\{1 + (k^2-1)\ln\mathcal{F}(\theta_1) + (k^2-1)\ln\mathcal{F}(\theta_2) + O((k^2-1)^2)\right\} \\
&= \frac{1}{2n}\int_0^{2n\pi} d\theta_2 \int_0^{2n\pi} d\theta_1 \left|2n\sin\frac{\theta_2-\theta_1}{2n}\right|^{-2k^2} \times \\
&\qquad \times \left\{1 + 2(k^2-1)\ln\mathcal{F}(\theta_2) + O\left((k^2-1)^2\right)\right\} \\
&= \frac{1}{2n}\int_0^{2n\pi} d\theta \left|2n\sin\frac{\theta}{2n}\right|^{-2k^2} \int_0^{2n\pi} d\theta' \left\{1 + 2(k^2-1)\ln\mathcal{F}(\theta')\right\} + O\left((k^2-1)^2\right) \\
&= \int_0^{2n\pi} d\theta \left|2n\sin\frac{\theta}{2n}\right|^{-2k^2} \int_{-\pi/2}^{\pi/2} d\theta' \left\{1 + 2(k^2-1)\ln\mathcal{F}(\theta')\right\} + O((k^2-1)^2)\,, \qquad (4.34)
\end{aligned}$$

where we used the periodicity of $\mathcal{F}(\theta)$ in the last step. Note that the two integrals factorize in the last line and the integral over θ' is independent of n. The integral over θ does not converge near the on-shell point $k^2 \simeq 1$. That is why we needed to introduce the regularization ϵ. In the momentum region $k^2 \simeq 1$, the divergence is coming only from the most singular part of the propagator which is independent of details of the Riemann surface and thus independent of n. Therefore, the divergent part cancels when we compute $K(k^2)$ through (4.33). Since the divergent part can be taken to be analytic in k^2, the finite part can be obtained by analytic continuation from the region $\mathrm{Re}(k^2) < 1/2$ where the integral converges.[9] It can be expressed in terms of the beta function as we see in (B.16) and vanishes when $k^2 = 1$:

$$\begin{aligned}
\int_0^{2n\pi} d\theta \left|2n\sin\frac{\theta}{2n}\right|^{-2k^2} &= (2n)^{-2k^2+1} B\left(\frac{1}{2}-k^2, \frac{1}{2}\right) \\
&= \frac{\pi}{n}(k^2-1) + O\left((k^2-1)^2\right). \qquad (4.35)
\end{aligned}$$

Therefore, $K(n, k^2)$ is given by

$$K(n,k^2) = \frac{\pi^2}{n}(k^2-1) + O\left((k^2-1)^2\right), \qquad (4.36)$$

and $K(k^2)$ is

$$K(k^2) = \frac{\pi^2}{3}(k^2-1) + O\left((k^2-1)^2\right). \qquad (4.37)$$

A more careful calculation using point-splitting regularization given in appendix B reproduces the same result (4.37) in the limit $\epsilon \to 0$ if $\mathcal{F}(\theta)$ is not too singular. Note that the coefficient in front of $k^2 - 1$ is independent of the conformal factor $\mathcal{F}(\theta)$. This is important for the consistency: since the on-shell cubic interaction does not depend on $\mathcal{F}(\theta)$, the on-shell three-tachyon coupling would depend on $\mathcal{F}(\theta)$ if this coefficient depended on $\mathcal{F}(\theta)$.

[9] I would like to thank Takuya Okuda for the discussion on this point. The explicit form of the singular part when we use point-splitting regularization is given in terms of the incomplete beta function in (B.16).

The tachyon field is therefore canonically normalized as follows:

$$\widehat{T}(k) = \left(\frac{\mathcal{K}\pi^2}{3}\right)^{1/2} T(k)\,. \tag{4.38}$$

4.3 Three-tachyon coupling and the D-brane tension

The tachyon cubic term is given by

$$\begin{aligned} S^{(3)} \;=\; &-\langle\Psi_g|\mathcal{Q}|\Psi_g\rangle \Big[\langle \mathit{3}|\mathit{0}\rangle + \langle \mathit{2}|\mathit{1}\rangle - \langle \mathit{3}|\mathit{0} * \mathit{0}\rangle - \\ &\qquad - \langle \mathit{2}|\mathit{1} * \mathit{0}\rangle - \langle \mathit{2}|\mathit{0} * \mathit{1}\rangle - \frac{1}{3}\langle \mathit{1}|\mathit{1} * \mathit{1}\rangle\Big]. \end{aligned} \tag{4.39}$$

This takes the following form:

$$S^{(3)} = -\frac{\mathcal{K}}{3}(2\pi)^{26}\int d^{26}k_1 d^{26}k_2 d^{26}k_3 \delta(k_1+k_2+k_3)\; T(k_1)T(k_2)T(k_3)V(k_1,k_2,k_3)\,. \tag{4.40}$$

We denote the on-shell value of $V(k_1,k_2,k_3)$ by V, which consists of the contributions from $\langle \mathit{3}|\mathit{0}\rangle$, $\langle \mathit{2}|\mathit{1}\rangle$, $\langle \mathit{3}|\mathit{0} * \mathit{0}\rangle$, $\langle \mathit{2}|\mathit{1} * \mathit{0}\rangle$, $\langle \mathit{2}|\mathit{0} * \mathit{1}\rangle$, and $\langle \mathit{1}|\mathit{1} * \mathit{1}\rangle$:

$$-\frac{1}{3}V = -\frac{1}{3}V(k_1,k_2,k_3)\bigg|_{k_1^2=k_2^2=k_3^2=1} = V_{30}+V_{21}-V_{300}-V_{210}-V_{201}-\frac{1}{3}V_{111}\,. \tag{4.41}$$

The contributions from $\langle \mathit{3}|\mathit{0}\rangle - \langle \mathit{3}|\mathit{0} * \mathit{0}\rangle$ and $\langle \mathit{2}|\mathit{1}\rangle - \langle \mathit{2}|\mathit{1} * \mathit{0}\rangle - \langle \mathit{2}|\mathit{0} * \mathit{1}\rangle$ might be expected to vanish if we naively use the equations of motion for $|\Xi_m\rangle$ and $|\chi_T(k)\rangle$. As can be anticipated from our experience in subsection 4.1, however, the calculations using point-splitting regularization presented in appendix D show that they do not vanish in the large-n limit:

$$V_{30}-V_{300}\neq 0\,, \qquad V_{21}-V_{210}-V_{201}\neq 0\,. \tag{4.42}$$

We find, however, a surprising cancellation between the two expressions so that the sum turns out to vanish in the large-n limit:

$$V_{30}+V_{21}-V_{300}-V_{210}-V_{201} = o(\epsilon)\,, \tag{4.43}$$

where we denote terms which vanish in the limit $\epsilon\to 0$ by $o(\epsilon)$. We will use this notation throughout the rest of the paper.[10] Therefore, only V_{111} contributes to V:

$$\frac{1}{3}V = \frac{1}{3}V_{111} + o(\epsilon)\,. \tag{4.44}$$

Let us calculate V_{111}. Since

$$k_1\cdot k_2 = \frac{1}{2}(k_1+k_2)^2 - \frac{1}{2}k_1^2 - \frac{1}{2}k_2^2 = -\frac{1}{2}\,, \tag{4.45}$$

[10] We distinguish $o(\epsilon)$ from $O(\epsilon)$. The latter denotes terms of order ϵ.

when $k_1^2 = k_2^2 = k_3^2 = 1$ and $k_1 + k_2 + k_3 = 0$, and similarly $k_2 \cdot k_3 = k_3 \cdot k_1 = -1/2$, V_{111} is given by

$$\begin{aligned} V_{111} = & \int_{\pi/2+\epsilon/2}^{3\pi/2-\epsilon/2} d\theta_3 \int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-3\pi/2+\epsilon/2}^{-\pi/2-\epsilon/2} d\theta_1 \times \\ & \times \left| 3\sin\frac{\theta_2-\theta_1}{3} \right|^{-1} \left| 3\sin\frac{\theta_3-\theta_1}{3} \right|^{-1} \left| 3\sin\frac{\theta_3-\theta_2}{3} \right|^{-1} . \end{aligned} \tag{4.46}$$

It turns out that V_{111} is finite in the limit $\epsilon \to 0$ so that we can set ϵ to zero. We present the calculation of the resulting integral in appendix E and the result is

$$\int_{\pi/2}^{3\pi/2} d\theta_3 \int_{-\pi/2}^{\pi/2} d\theta_2 \int_{-3\pi/2}^{-\pi/2} d\theta_1 \left| 3\sin\frac{\theta_2-\theta_1}{3} \right|^{-1} \left| 3\sin\frac{\theta_3-\theta_1}{3} \right|^{-1} \left| 3\sin\frac{\theta_3-\theta_2}{3} \right|^{-1} = \frac{\pi^2}{3} . \tag{4.47}$$

Therefore, we have

$$V = \frac{\pi^2}{3} + o(\epsilon) . \tag{4.48}$$

The on-shell three-tachyon coupling g_T is defined by the on-shell value of the cubic interaction when we express $S^{(3)}$ in terms of the canonically normalized tachyon $\widehat{T}(k)$. Namely, g_T is given by

$$g_T = \widehat{V}(k_1, k_2, k_3)\Big|_{k_1^2=k_2^2=k_3^2=1} , \tag{4.49}$$

where $\widehat{V}(k_1, k_2, k_3)$ is defined by

$$S^{(3)} = -\frac{1}{3}(2\pi)^{26} \int d^{26}k_1\, d^{26}k_2\, d^{26}k_3 \delta(k_1+k_2+k_3) \widehat{T}(k_1)\widehat{T}(k_2)\widehat{T}(k_3)\widehat{V}(k_1,k_2,k_3) . \tag{4.50}$$

From the relation (4.38) between $T(k)$ and $\widehat{T}(k)$, and the on-shell cubic interaction (4.48), g_T is given by

$$g_T = \left(\frac{\mathcal{K}\pi^2}{3} \right)^{-1/2} . \tag{4.51}$$

The tension of a single D25-brane T_{25} predicted by the three-tachyon coupling g_T through (1.1) is given by

$$T_{25} = \frac{1}{2\pi^2 g_T^2} = \frac{\mathcal{K}}{6} . \tag{4.52}$$

On the other hand, the energy density $\mathcal{E}_c$ of the classical solution $|\Xi_m\rangle \otimes |\Psi_g\rangle$ is given by

$$\begin{aligned} \int d^{26}x\, \mathcal{E}_c &= \frac{1}{2} \langle \Xi_m | \Xi_m \rangle \langle \Psi_g | \mathcal{Q} | \Psi_g \rangle + \frac{1}{3} \langle \Xi_m | \Xi_m * \Xi_m \rangle \langle \Psi_g | \Psi_g * \Psi_g \rangle \\ &= \frac{1}{6} \langle \Xi_m | \Xi_m \rangle \langle \Psi_g | \mathcal{Q} | \Psi_g \rangle = \int d^{26}x\, \frac{\mathcal{K}}{6} . \end{aligned} \tag{4.53}$$

This is in perfect agreement with the interpretation that the configuration $|\Xi_m\rangle \otimes |\Psi_g\rangle$ describes a single D25-brane:

$$T_{25} = \mathcal{E}_c . \tag{4.54}$$

This is our main result in this paper.

The calculations in appendix D are so complicated that it would be difficult to confirm that the cancellation (4.43) does not depend on details of the regularization. We will give a calculation in subsection 5.4 which might be regarded as a piece of evidence that the result (4.48) is not sensitive to details of the regularization.

5. Discussion

In this section we discuss some problematic issues of our formulation. We then discuss a reformulation of our description which is rather close to BSFT, and end with future directions.

5.1 Is the sliver state a classical solution?

We found in subsections 4.1 and 4.3 that the sliver state $|\Xi_m\rangle$ and the linearized on-shell open string states $|\chi_{\varphi_i}(k)\rangle$ do not satisfy their equations of motion when they are contracted with the class of states $|\{\varphi_i\}\rangle$ (2.12),

$$\begin{aligned} \langle\{\varphi_i\}|\Xi_m\rangle &\neq \langle\{\varphi_i\}|\Xi_m * \Xi_m\rangle\,, \\ \langle\{\varphi_i\}|\chi_{\varphi_i}(k)\rangle &\neq \langle\{\varphi_i\}|\chi_{\varphi_i}(k) * \Xi_m\rangle + \langle\{\varphi_i\}|\Xi_m * \chi_{\varphi_i}(k)\rangle\,, \end{aligned} \tag{5.1}$$

while they satisfy their equations of motion when contracted with an arbitrary state in the matter Fock space $|\phi\rangle$,

$$\begin{aligned} \langle\phi|\Xi_m\rangle &= \langle\phi|\Xi_m * \Xi_m\rangle\,, \\ \langle\phi|\chi_{\varphi_i}(k)\rangle &= \langle\phi|\chi_{\varphi_i}(k) * \Xi_m\rangle + \langle\phi|\Xi_m * \chi_{\varphi_i}(k)\rangle\,. \end{aligned} \tag{5.2}$$

Does this mean that the sliver state is not a classical solution of VSFT and that we are expanding the action around an inappropriate configuration?

Let us see if the linear terms of the open string fields $\{\varphi_i\}$ vanish when we express the VSFT action in terms of them. The part of the VSFT action which is linear in $\{\varphi_i\}$ is given by

$$S^{(1)} = -\langle\Psi_g|\mathcal{Q}|\Psi_g\rangle \int d^{26}k\varphi_i(k)\left[\langle\chi_{\varphi_i}(k)|\Xi_m\rangle - \langle\chi_{\varphi_i}(k)|\Xi_m * \Xi_m\rangle\right]. \tag{5.3}$$

The inner products $\langle\chi_{\varphi_i}(k)|\Xi_m\rangle$ and $\langle\chi_{\varphi_i}(k)|\Xi_m * \Xi_m\rangle$ are expressed in terms of one-point functions of the vertex operator $\mathcal{O}_{\varphi_i(k)}$. It is obvious from momentum conservation that the one-point functions vanish for a nonzero momentum. The vertex operators for zero-momentum open string fields other than the tachyon have a nonzero conformal dimension so that their one-point functions vanish. Therefore, the tachyon is the only dangerous field which may have a nonvanishing linear term.

Let us therefore calculate the tachyon potential $V(T)$ in our description of the tachyon field $|e^{-T}\rangle$. For a constant tachyon field $T(x) = T$, the factor inserted in $|e^{-T}\rangle$ in the cone representation is given by

$$\exp\left[-T\int_{-(n-1)\pi/2}^{(n-1)\pi/2} d\theta\,\mathcal{F}_0(\theta)^{-1}\right] = e^{-aT}\,, \tag{5.4}$$

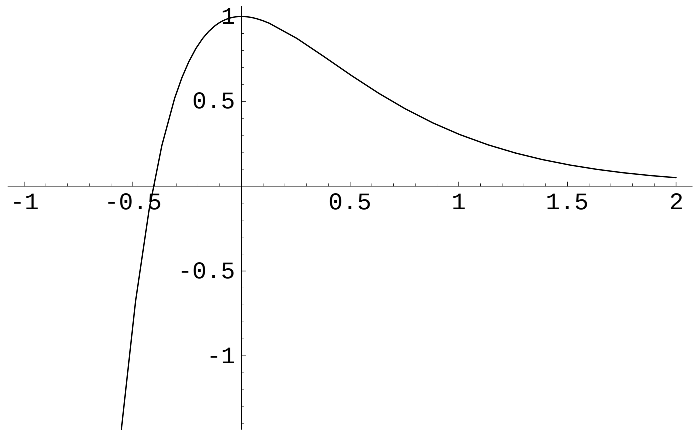

Figure 1: The tachyon potential $V(T)$. The horizontal axis is aT and the vertical axis is $V(T)/T_{25}$. The sliver state corresponds to $T=0$ and the tachyon vacuum corresponds to $T=\infty$.

where the value of a constant a,

$$a = \int_{-(n-1)\pi/2}^{(n-1)\pi/2} d\theta\, \mathcal{F}_0(\theta)^{-1}\,, \tag{5.5}$$

depends on the choice of the Riemann surface Σ_n where the off-shell tachyon is defined. The tachyon potential $V(T)$ is easily calculated from (4.4) and the result is

$$V(T) = \mathcal{K}\left(\frac{1}{2}e^{-2aT} - \frac{1}{3}e^{-3aT}\right) = 3T_{25}e^{-2aT} - 2T_{25}e^{-3aT}\,, \tag{5.6}$$

where we used (4.52). The shape of $V(T)$ is given in figure 1. The linear term vanishes at $T=0$,

$$\left.\frac{dV(T)}{dT}\right|_{T=0} = 0\,. \tag{5.7}$$

The tachyon vacuum corresponds to $T=\infty$ where $V(T)=0$, and the potential height at $T=0$ is exactly the same as the D25-brane tension as we calculated in subsection 4.3:

$$V(0) - V(\infty) = T_{25}\,. \tag{5.8}$$

To summarize, the linear terms of the open string fields, including the tachyon, vanish at the configuration where $\varphi_i = 0$, which corresponds to the sliver state:

$$\left.\frac{\delta S[\{\varphi_i\}]}{\delta\varphi_i}\right|_{\varphi_i=0} = 0\,. \tag{5.9}$$

In this sense, we can regard the sliver state as a classical solution. However, we do not completely understand whether the breakdown of the equations of motion in the form of (5.1) is problematic or not.

5.2 The large-n limit and renormalization

The primary goal of the present paper was to study if VSFT describes the ordinary perturbative dynamics of open strings based on our proposal. For this purpose, we calculated the kinetic term of the tachyon near its mass shell and the on-shell three-tachyon coupling in section 4. We found that all the divergences we encountered in these calculations canceled and these quantities stay finite in the large-n limit. However, the tachyon state $|e^{-T}\rangle$ itself, or more general states $|\{\varphi_i\}\rangle$, do not seem to have the large-n limit.

As we have seen in section 4, singularities in the large-n limit become short-distance singularities such as $1/\epsilon$ or $\ln\epsilon$ when we calculate correlation functions on the unit disk or the cone with an angle 3π. The open string fields $\{\varphi_i\}$ correspond to bare coupling constants in the open string sigma model and we may need to renormalize them appropriately in the large-n limit to make our description more well-defined.

It might be useful to notice that there were two different types of divergences in the calculations in section 4. Take $\langle \mathit{2}|\mathit{0}\rangle$ as an example. The $1/\epsilon$ divergence of $K_{20}(1)$ in (4.25) comes from the contribution where the two vertex operators become close in the bulk of the integration region. On the other hand, the $\ln\epsilon$ divergence of $K_{20}(1)$ in (4.25) can be regarded as a boundary effect of the integration region. To understand this, the following analogy might be helpful. The inner product $\langle \mathit{2}|\mathit{0}\rangle$ is the quadratic part of $\langle\{\varphi_i\}|\Xi_m\rangle$ in the expansion of $T(k)$ when we turn on only the tachyon field. The boundary interaction

$$\exp\left[-\int dt\int d^{26}k\sum_i \varphi_i(k)\mathcal{O}_{\varphi_i(k)}(t)\right] \tag{5.10}$$

in (2.12) is introduced only on a part of the boundary in $\langle\{\varphi_i\}|\Xi_m\rangle$. The inner product $\langle\{\varphi_i\}|\Xi_m\rangle$ is therefore analogous to an open Wilson line in ordinary gauge theory in this respect. On the other hand, the boundary interaction is introduced on the whole boundary in the inner products $\langle\{\varphi_i\}|\{\varphi_i\}\rangle$ and $\langle\{\varphi_i\}|\{\varphi_i\}*\{\varphi_i\}\rangle$ appearing in the VSFT action (4.1). These inner products are analogous to closed Wilson loops. When we expand the Wilson-loop-like inner products $\langle\{\varphi_i\}|\{\varphi_i\}\rangle$ and $\langle\{\varphi_i\}|\{\varphi_i\}*\{\varphi_i\}\rangle$ as we did in (4.7), the combinations $\langle \mathit{1}|\mathit{1}\rangle/2+\langle \mathit{2}|\mathit{0}\rangle$ and $\langle \mathit{1}|\mathit{1}*\mathit{0}\rangle+\langle \mathit{2}|\mathit{0}*\mathit{0}\rangle$ appear, and the $\ln\epsilon$ divergence is absent from $K_{11}(1)/2+K_{20}(1)$ and $K_{110}(1)+K_{200}(1)$ corresponding to these combinations of inner products.[11] We can generally expect that the divergence coming from the boundary of the boundary interaction cancels in calculations of Wilson-loop-like quantities such as the VSFT action.

The other type of divergence such as $1/\epsilon$ in $K_{20}(1)$ is familiar in the open string sigma model. We will be able to handle such divergences by a conventional renormalization procedure, at least for renormalizable boundary interactions, and the situation is similar to that of BSFT. Note also that a multiplicative renormalization of the tachyon field will not change the on-shell three-tachyon coupling constant g_T we calculated in section 4.

On the other hand, we do not know how to handle the divergence coming from the boundary of the boundary interaction. If this class of divergence remains in a physically

[11] The absence of the $\ln\epsilon$ divergence is more transparent in the calculations of $\widetilde{K}_{11}/2+\widetilde{K}_{20}$ and $\widetilde{K}_{110}+\widetilde{K}_{200}$ in appendix C.

relevant calculation, it can be a problem of our formulation, although we have not encountered such situations so far and we do not expect such problems in the calculation of the VSFT action as we mentioned before. Incidentally, inner products of general states with a state in the matter Fock space, $\langle\{\varphi_i\}|\phi\rangle$, become Wilson-loop-like in the large-n limit. It seems therefore possible to make them well-defined by a conventional renormalization procedure for renormalizable interactions, although it is not clear if the inner products $\langle\{\varphi_i\}|\phi\rangle$ are really physically relevant in VSFT.

5.3 Off-shell definitions

In section 2 we mentioned an ambiguity in our off-shell definition of open string fields. If we need to renormalize the open string fields as we discussed in the previous subsection, the choice of the renormalization scheme will be another source of off-shell ambiguity.[12] It is important for the consistency of our formulation that such off-shell ambiguity does not affect physically relevant quantities. We found, for example, that the relation between the D25-brane tension T_{25} and the on-shell three-tachyon coupling constant g_T derived in section 4 was independent of the ambiguity coming from $\mathcal{F}_0(\theta)$ in (2.11). If the physics is really independent of the off-shell ambiguity, we can in principle choose any off-shell definition. However, an inappropriate choice might cause a singular behavior of the off-shell fields.

If we take the Riemann surface Σ_n to be a cone with an angle $n\pi$, which corresponds to $\mathcal{F}_0(\theta) = 1$ in (2.11) for the tachyon, the tachyon kinetic term can be calculated exactly for $\mathrm{Re}(k^2) < 1/2$. From the calculation in subsection 4.2, it is given by

$$\begin{aligned} \frac{1}{2}K(k^2) &= \frac{\pi}{2}\left(\frac{1}{n-1}\right)^{2k^2-2}\left(2^{-2k^2+1} - 3^{-2k^2+1}\right) B\left(\frac{1}{2} - k^2, \frac{1}{2}\right) \\ &= \frac{\pi}{2}\left(\frac{1}{n-1}\right)^{2k^2-2}\left(2^{-2k^2+1} - 3^{-2k^2+1}\right) \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2} - k^2\right)}{\Gamma(1-k^2)}\,. \end{aligned} \tag{5.11}$$

We may renormalize the tachyon field to absorb the n-dependent factor in (5.11) coming from (4.15), which is not relevant to the present discussion. If we assume the analyticity in k^2, we can define $K(k^2)$ for all k^2 by analytic continuation. We then note that the kinetic term (5.11) vanishes not only at $k^2 = 1$ but also at any positive integer $k^2 = 1, 2, 3, \ldots$ The kinetic term is also singular at $k^2 = 3/2, 5/2, \ldots$, where it diverges. If this implies the existence of an infinite number of tachyons, the theory will definitely be pathological, and it can be a problem of our formulation.

We have argued that it is universal that the tachyon kinetic term vanishes when $k^2 = 1$, but it is not clear if other zeros of (5.11) at $k^2 = 2, 3, \ldots$ are also universal. The singular behavior of (5.11) for higher k^2 might be an artifact of an inappropriate off-shell definition. In fact, when we increase the momentum k in the calculation of $K(k^2)$ using point-splitting regularization, the next-to-leading singularity in (B.18) becomes divergent at $k^2 = 3/2$,

[12]These two ambiguities might not be independent. The conformal factor $\mathcal{F}_0(\theta)^{k^2-1}$ in (2.11) has an implicit dependence on n through the choice of the Riemann surface Σ_n, and the θ-independent part of this factor looks similar to a multiplicative renormalization of $T(k)$.

and it is n-dependent. The divergence in $K(k^2)$ will no longer cancel and the tachyon field beyond $k^2 = 3/2$ seems to depend strongly on the off-shell ambiguity. It is not clear if the tachyon kinetic term universally vanishes at $k^2 = 2, 3, \ldots$ We cannot claim anything definite about this issue for now, but we hope that the expression (5.11) corresponds to a singular definition of the tachyon and there is a better class of off-shell definitions.

5.4 BSFT-like reformulation

So far we have found that when we express the VSFT action in terms of the open string fields $\{\varphi_i\}$ based on our proposal (2.12),

1. the linear terms vanish,
2. the kinetic terms vanish when the fields $\{\varphi_i\}$ satisfy the physical state conditions,
3. and the relation between the D25-brane tension T_{25} and the on-shell three-tachyon coupling g_T given by (1.1) is correctly reproduced.

Although we started from VSFT, most of the calculations in the present paper are reminiscent of those of boundary string field theory (BSFT) [12, 13, 14, 15, 16, 31, 32].[13] Furthermore, the problems we discussed in subsections 5.1 and 5.2 seem to suggest a BSFT-like reformulation of our description. In fact, all the results we just mentioned are effectively reproduced by the following BSFT-like action:

$$\begin{aligned} S[\{\varphi_i\}] = & -3T_{25} \left\langle \exp\left[-\int d^{26}k \sum_i \varphi_i(k) \int_0^{2\pi} d\theta \; \mathcal{O}_{\varphi_i(k)}(e^{i\theta}) \right] \right\rangle_{\text{disk}} + \\ & + 2T_{25} \left\langle \exp\left[-\int d^{26}k \sum_i \varphi_i(k) \int_0^{3\pi} d\theta \; \mathcal{O}_{\varphi_i(k)}(e^{i\theta}) \right] \right\rangle_{3\pi}, \end{aligned} \tag{5.12}$$

where the correlation functions are normalized as

$$\langle 1 \rangle_{\text{disk}} = \int d^{26}x \,, \qquad \langle 1 \rangle_{3\pi} = \int d^{26}x \,, \tag{5.13}$$

and we assume an appropriate regularization and renormalization scheme. As we have discussed, the off-shell definition for $\{\varphi_i\}$ is not unique. For example, we can define off-shell tachyon field taking the ambiguity into account as follows:

$$\begin{aligned} S[T(k)] = & -3T_{25} \left\langle \exp\left[-\int d^{26}k \, T(k) \int_0^{2\pi} d\theta \; \mathcal{F}(\theta)^{k^2-1} e^{ikX}(e^{i\theta}) \right] \right\rangle_{\text{disk}} + \\ & + 2T_{25} \left\langle \exp\left[-\int d^{26}k \, T(k) \int_0^{3\pi} d\theta \mathcal{F}(\theta)^{k^2-1} e^{ikX}(e^{i\theta}) \right] \right\rangle_{3\pi}, \end{aligned} \tag{5.14}$$

where we extended the definition of $\mathcal{F}(\theta)$ from $-\pi/2 \le \theta \le \pi/2$ to all θ through $\mathcal{F}(\theta+\pi) = \mathcal{F}(\theta)$ as we did in subsection 4.2. We can easily see that the linear terms of $\{\varphi_i\}$ vanish

[13] In particular, similar calculations can be found in appendix A of [32]. Related calculations can also be found in a different context in [33].

in the action (5.12) and the tachyon potential is given by (5.6). If we use point-splitting regularization, we can also show that the kinetic terms for $\{\varphi_i\}$ vanish when $\mathcal{O}_{\varphi_i(k)}$ is primary with conformal dimension one, and the calculations for the tachyon kinetic term near its mass shell are simpler than those in subsection 4.2 and appendix B, and give the same result.

The calculation for the on-shell three-tachyon coupling is remarkably simpler than the VSFT calculation of appendices D and E. Before regularizing it, the on-shell cubic interaction V defined by (4.41) can be written as follows:

$$-\frac{1}{3}V = V(1) - V\left(\frac{3}{2}\right), \tag{5.15}$$

where $V(n)$ is defined by

$$\begin{aligned} V(n) &\equiv \frac{1}{2n}\frac{1}{3!}\int_0^{2n\pi} d\theta_3 \int_0^{2n\pi} d\theta_2 \int_0^{2n\pi} d\theta_1 \times \\ &\quad \times \left|2n\sin\frac{\theta_1-\theta_2}{2n}\right|^{-1} \left|2n\sin\frac{\theta_2-\theta_3}{2n}\right|^{-1} \left|2n\sin\frac{\theta_3-\theta_1}{2n}\right|^{-1} \\ &= \frac{\pi}{6}\int_0^{2n\pi} d\theta_2 \int_0^{2n\pi} d\theta_1 \left|2n\sin\frac{\theta_1-\theta_2}{2n}\right|^{-1} \left|2n\sin\frac{\theta_1}{2n}\right|^{-1} \left|2n\sin\frac{\theta_2}{2n}\right|^{-1}. \end{aligned} \tag{5.16}$$

If we use point-splitting regularization, $V(n)$ is regularized and calculated in the following way:

$$\begin{aligned} V(n) &= \frac{\pi}{3}\int_{2\epsilon}^{2n\pi-\epsilon} d\theta_2 \int_{\epsilon}^{\theta_2-\epsilon} d\theta_1 \left(2n\sin\frac{\theta_2-\theta_1}{2n}\right)^{-1} \left(2n\sin\frac{\theta_1}{2n}\right)^{-1} \left(2n\sin\frac{\theta_2}{2n}\right)^{-1} \\ &= \frac{2\pi}{3(2n)^2}\int_{2\epsilon}^{2n\pi-\epsilon} d\theta_2 \left(\sin\frac{\theta_2}{2n}\right)^{-2} \ln\frac{\sin\frac{\theta_2-\epsilon}{2n}}{\sin\frac{\epsilon}{2n}} \\ &= -\frac{\pi^2}{3n}\left(1-\frac{3\epsilon}{2n\pi}\right) + \frac{\pi}{n}\cot\frac{\epsilon}{2n}\ln\frac{\sin\frac{\epsilon}{n}}{\sin\frac{\epsilon}{2n}} \\ &= \frac{2\pi\ln 2}{\epsilon} - \frac{\pi^2}{3n} + O(\epsilon)\,. \end{aligned} \tag{5.17}$$

Therefore, V is given by

$$\frac{1}{3}V = \frac{\pi^2}{9} + O(\epsilon)\,. \tag{5.18}$$

This coincides with the result (4.48), and therefore gives the relation between T_{25} and g_T (1.1) correctly. This calculation seems to indicate that the complication in the calculations of appendix D is due to the existence of the boundary of the boundary interaction and the final result for V, (4.48), is not sensitive to details of the regularization.

Since this BSFT-like formulation is simpler than VSFT for this kind of calculation, it might be useful to test if other aspects of string perturbation theory are correctly reproduced. It would also be interesting to explore if the action (5.12) itself defines a new string field theory. The problem regarding nonrenormalizable boundary interactions would be taken over from that of ordinary BSFT. However, we expect some cancellation of the divergences between the two terms in (5.12) as we have seen in section 4 and appendices B and D. It would be important to understand the structure of the divergences in (5.12), or in (4.1).

Interestingly, the results we listed at the beginning of this subsection are also reproduced by a class of BSFT-like actions[14] such as

$$S_{n,m}[\{\varphi_i\}] = -\frac{\mathcal{K}}{2n} \left\langle \exp\left[-\int d^{26}k \sum_i \varphi_i(k) \int_0^{2n\pi} d\theta\, \mathcal{O}_{\varphi_i(k)}(e^{i\theta})\right]\right\rangle_{2n\pi} + \\ +\frac{\mathcal{K}}{2m} \left\langle \exp\left[-\int d^{26}k \sum_i \varphi_i(k) \int_0^{2m\pi} d\theta\, \mathcal{O}_{\varphi_i(k)}(e^{i\theta})\right]\right\rangle_{2m\pi}, \quad (5.19)$$

with $n < m$,[15] or a linear combination of $S_{n,m}$'s. The action (5.12) corresponds to the case where $n/m = 2/3$. It would be important to understand whether this value, which is inherited from Witten's open string field theory [17], has any special meaning or not.

If we take the large-m limit while n is kept finite ($n/m \to 0$) after point-splitting regularization, the second term in (5.19) just subtracts the divergences and does not contribute to the finite terms as far as the calculations we have done so far are concerned. In this limit, the action may be related to the (renormalized) partition function with the boundary interaction [34]–[38].[16] The tachyon potential, however, becomes singular in this limit.

If we choose $m = n + a$ and take n to be large while a is kept finite ($n/m \to 1$), the two terms in (5.19) become almost the same. A conformal transformation which maps one Riemann surface to the other becomes infinitesimal so that the resulting action may be related to the BSFT [12] where the anticommutator of the BRST charge with the vertex operator is inserted. Let us calculate the tachyon potential in this limit. Under an appropriate normalization for the constant tachyon field T, it is given by

$$V(T) = \frac{\mathcal{K}}{2n} e^{-nT} - \frac{\mathcal{K}}{2(n+a)} e^{-(n+a)T}\,. \quad (5.20)$$

Since the D25-brane tension T_{25} is given by

$$T_{25} = \frac{\mathcal{K}}{2n} - \frac{\mathcal{K}}{2(n+a)} = \frac{\mathcal{K}a}{2n(n+a)}\,, \quad (5.21)$$

the tachyon potential is normalized as follows:

$$\frac{V(T)}{T_{25}} = \frac{n+a}{a} e^{-nT} - \frac{n}{a} e^{-(n+a)T}\,. \quad (5.22)$$

We can take the large-n limit if we redefine the tachyon as $T \to T/n$ and the resulting potential is given by

$$\frac{V(T)}{T_{25}} = (T+1)e^{-T}\,. \quad (5.23)$$

This coincides with the tachyon potential in BSFT [13, 31, 32]. It would be interesting to learn more about the relation between the action (5.19) and BSFT.

[14]I would like to thank Takuya Okuda for suggesting this possibility.

[15]The ratio n/m must be rational if we want to incorporate the off-shell ambiguity we mentioned in section 2.

[16]See also a recent work [39].

5.5 Ghost solution

We have concentrated on the matter part of VSFT assuming the existence of the universal solution in the ghost part. Actually, we implicitly assume more about the ghost solution.

In the CFT formulation of string field theory [24], we implicitly use the generalized gluing and resmoothing theorem [25]. As is emphasized in [40], the theorem holds only when the total central charge vanishes. Furthermore, we have to make the same conformal transformation for both matter and ghost sectors. Otherwise a conformal anomaly effectively occurs even when the total central charge vanishes.

Since all the states we considered in this paper were defined as the large-n limit of wedge states with some operator insertions, the universal ghost solution also has to share this property. Namely, we assume the existence of a series of purely ghost operators $\mathcal{Q}(n)$ and ghost states $|\Psi_g(n)\rangle$ labeled by n satisfying the following conditions:

1. the ghost operators $\mathcal{Q}(n)$ have vanishing cohomology, at least in the large-n limit,
2. the ghost states $|\Psi_g(n)\rangle$ take the form of wedge states labeled by n with some ghost insertions,
3. they solve the ghost equation of motion in the large-n limit:
$$\lim_{n\to\infty} \langle\Psi_g(n)|\mathcal{Q}(n)|\Psi_g(n)\rangle + \lim_{n\to\infty} \langle\Psi_g(n)|\Psi_g(n) * \Psi_g(n)\rangle = 0\,, \tag{5.24}$$
4. and they give a finite D-brane tension:
$$\lim_{n\to\infty} \langle n|n\rangle\, \langle\Psi_g(n)|\mathcal{Q}(n)|\Psi_g(n)\rangle = \int d^{26}x\, \mathcal{K}\,, \tag{5.25}$$
where $\mathcal{K}$ is finite.

Can we find such operators $\mathcal{Q}(n)$ and states $|\Psi_g(n)\rangle$? The ghost solution found by Hata and Kawano [6] turned out to be described as the sliver state of the twisted ghost CFT [41]. Unfortunately, the total central charge does not vanish when we twist the ghost CFT. However, we can show that a class of wedge states in the twisted ghost CFT are described by wedge states in the untwisted ghost CFT with some operator insertions [42]. We assume that such ghost states satisfy the conditions (5.24) and (5.25) under an appropriate regularization. We hope to report on this issue in a future work [42].

5.6 Future directions

In this paper we used the CFT formulation [24, 25] of string field theory. In the development of VSFT the interplay among various formulations has played an important role. The D25-brane solution in VSFT was first studied in the operator formalism [43, 6]. The technology in the operator formalism of string field theory [19]–[23] is developing rapidly based on the spectroscopy of the Neumann coefficient matrices [44]–[49]. It is used for analytical proofs of conjectured equivalence between results from the CFT descriptions and corresponding ones in the operator formalism [46, 50, 51, 52]. The half-string or split-string picture [53]–[56] is particularly useful when we consider systems of multiple D-branes. It is further related to a recent reformulation in terms of noncommutative field theory [57, 58]. It would be important to study relations of our description to these other formulations for future investigations on generalizations to non-abelian cases, the role of the gauge invariance, and so on. An approach to non-abelian structures from the viewpoint of BSFT such as [59] may also be useful.

We used wedge states to formulate our description of the open string fields on a D25-brane by making use of their simple star algebra [18]. On the other hand, other star algebra projectors such as the butterfly state were also found and studied [60, 61]. It would be interesting to consider the description of open string fields for other surface states and see if the results such as the ones we listed at the beginning of subsection 5.4 are independent of the choice of the surfaces.

Finally, it would be very important to understand relations of our formulation to Witten's cubic string field theory [17] as well as further relations to BSFT [12]. Witten's cubic string field theory is based on the BRST quantization while our formulation is reminiscent of the old covariant quantization. We hope that our formulation will give some insights into dictionaries between Witten's cubic string field theory, BSFT, and VSFT.

Acknowledgments

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A. The relation between T_{25} and g_T

The inverse of the D25-brane tension T_{25} and the square of the on-shell three-tachyon coupling constant g_T are both proportional to the string coupling constant. The dimensionless quantity $\alpha'^3\, T_{25}\, g_T^2$ is therefore independent of the string coupling constant. Let us calculate it following the convention of [5].

The effective action for the tachyon is given in (6.5.16) of [5] by

$$S = \frac{1}{{g'_o}^2} \int d^{26}x \left[-\frac{1}{2}\partial_\mu T(x) \partial^\mu T(x) + \frac{1}{2\alpha'} T(x)^2 + \frac{1}{3}\sqrt{\frac{2}{\alpha'}}\, T(x)^3 \right]. \tag{A.1}$$

The tension of a Dp-brane T_p is given in (8.7.26) of [5] by

$$T_p^2 = \frac{\pi}{256\kappa^2} (4\pi^2\alpha')^{11-p}. \tag{A.2}$$

The relation between g'_o and κ is given in (8.7.28) of [5] by

$$\frac{4\pi\alpha' {g'_o}^2}{\kappa} = 2^{18}\pi^{25/2}\alpha'^6. \tag{A.3}$$

Therefore, the D25-brane tension T_{25} is expressed in terms of g'_o by

$$T_{25} = \frac{1}{4\pi^2\alpha'^2 {g'_o}^2} \,. \tag{A.4}$$

The on-shell three-tachyon coupling g_T is defined by[17]

$$S = \int d^{26}x \left[-\frac{1}{2}\partial_\mu \widehat{T}(x)\partial^\mu \widehat{T}(x) + \frac{1}{2\alpha'}\widehat{T}(x)^2 + \frac{1}{3} g_T \widehat{T}(x)^3 \right] . \tag{A.5}$$

Since the normalized tachyon field $\widehat{T}$ is related to T in (A.1) as

$$\widehat{T}(x) = \frac{T(x)}{g'_o} \,, \tag{A.6}$$

the relation between g_T and g'_o is given by

$$g_T = g'_o \sqrt{\frac{2}{\alpha'}} \,. \tag{A.7}$$

The D25-brane tension T_{25} is therefore expressed in terms of g_T as

$$T_{25} = \frac{1}{2\pi^2\alpha'^3 g_T^2} \,. \tag{A.8}$$

B. $K(k^2)$

We calculate $K(k^2)$ in (4.10) when k^2 is nearly on shell $k^2 \simeq 1$ to show (4.37) in this appendix. Let us begin with $K_{20}(k^2)$ (4.22) and $K_{200}(k^2)$ (4.23). Both take the following form:

$$K_{20/200}(n, k^2) = \int_{-\pi/2+3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi/2+\epsilon/2}^{\theta_2-\epsilon} d\theta_1 \, \mathcal{F}(\theta_1)^{k^2-1}\mathcal{F}(\theta_2)^{k^2-1} \left| 2n \sin\frac{\theta_2-\theta_1}{2n} \right|^{-2k^2} , \tag{B.1}$$

with $n = 1$ for K_{20} and $n = 3/2$ for K_{200}. Using the formula

$$\int_{-\pi/2+3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi/2+\epsilon/2}^{\theta_2-\epsilon} d\theta_1 \, f(\theta_1,\theta_2) = \int_{-\pi/2+\epsilon/2}^{\pi/2-3\epsilon/2} d\theta_2 \int_{\theta_2+\epsilon}^{\pi/2-\epsilon/2} d\theta_1 \, f(\theta_2,\theta_1) \tag{B.2}$$

[17] The normalized tachyon $\widehat{T}(x)$ here is related to $\widehat{T}(k)$ in subsection 4.2 as

$$\widehat{T}_{\rm Polchinski}(x) = -\widehat{T}_{\rm ours}(x)$$

with

$$\widehat{T}_{\rm ours}(x) = \int d^{26}k \; \widehat{T}(k) e^{ikX} \,.$$

for any function f with two variables, $K_{20/200}(n,k^2)$ is rewritten in the following way:

$$\begin{aligned}
K_{20/200}(n,k^2) &= \frac{1}{2}\int_{-\pi/2+3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi/2+\epsilon/2}^{\theta_2-\epsilon} d\theta_1\ \mathcal{F}(\theta_1)^{k^2-1}\mathcal{F}(\theta_2)^{k^2-1}\left|2n\sin\frac{\theta_2-\theta_1}{2n}\right|^{-2k^2} + \\
&\quad +\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-3\epsilon/2} d\theta_2 \int_{\theta_2+\epsilon}^{\pi/2-\epsilon/2} d\theta_1\ \mathcal{F}(\theta_1)^{k^2-1}\mathcal{F}(\theta_2)^{k^2-1}\left|2n\sin\frac{\theta_2-\theta_1}{2n}\right|^{-2k^2} \\
&= \frac{1}{2}\int_{-\pi/2+3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi/2+\epsilon/2}^{\theta_2-\epsilon} d\theta_1\ \left|2n\sin\frac{\theta_2-\theta_1}{2n}\right|^{-2k^2} \times \\
&\qquad \times\left\{1+(k^2-1)\ln\mathcal{F}(\theta_1)+(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\} + \\
&\quad +\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-3\epsilon/2} d\theta_2 \int_{\theta_2+\epsilon}^{\pi/2-\epsilon/2} d\theta_1\ \left|2n\sin\frac{\theta_2-\theta_1}{2n}\right|^{-2k^2} \times \\
&\qquad \times\left\{1+(k^2-1)\ln\mathcal{F}(\theta_1)+(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\} \\
&= \frac{1}{2}\int_{-\pi/2+3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi/2+\epsilon/2}^{\theta_2-\epsilon} d\theta_1\ \left|2n\sin\frac{\theta_2-\theta_1}{2n}\right|^{-2k^2} \times \\
&\qquad \times\left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\} + \\
&\quad +\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-3\epsilon/2} d\theta_2 \int_{\theta_2+\epsilon}^{\pi/2-\epsilon/2} d\theta_1\ \left|2n\sin\frac{\theta_2-\theta_1}{2n}\right|^{-2k^2} \times \\
&\qquad \times\left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\}.
\end{aligned} \tag{B.3}$$

Next we rewrite $K_{11}(k^2)$ (4.14) and $K_{110}(k^2)$ (4.17) similarly:

$$\begin{aligned}
K_{11}(k^2) &= \int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi}^{-\pi/2-\epsilon/2} d\theta_1\ \left|2\sin\frac{\theta_2-\theta_1}{2}\right|^{-2k^2} \times \\
&\qquad \times\left\{1+(k^2-1)\ln\mathcal{F}(\theta_1+\pi)+(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\} + \\
&\quad +\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\pi/2+\epsilon/2}^{\pi} d\theta_1\ \left|2\sin\frac{\theta_2-\theta_1}{2}\right|^{-2k^2} \times \\
&\qquad \times\left\{1+(k^2-1)\ln\mathcal{F}(\theta_1-\pi)+(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\} \\
&= \int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi}^{-\pi/2-\epsilon/2} d\theta_1\ \left|2\sin\frac{\theta_2-\theta_1}{2}\right|^{-2k^2} \times \\
&\qquad \times\left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\} + \\
&\quad +\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\pi/2+\epsilon/2}^{\pi} d\theta_1\ \left|2\sin\frac{\theta_2-\theta_1}{2}\right|^{-2k^2} \times \\
&\qquad \times\left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\},
\end{aligned} \tag{B.4}$$

and

$$\begin{aligned}
K_{110}(k^2) &= \int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-3\pi/2+\epsilon/2}^{-\pi/2-\epsilon/2} d\theta_1\ \mathcal{F}(\theta_1+\pi)^{k^2-1}\mathcal{F}(\theta_2)^{k^2-1}\left|3\sin\frac{\theta_2-\theta_1}{3}\right|^{-2k^2} \\
&= \frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-3\pi/2+\epsilon/2}^{-\pi/2-\epsilon/2} d\theta_1\ \mathcal{F}(\theta_1+\pi)^{k^2-1}\mathcal{F}(\theta_2)^{k^2-1}\left|3\sin\frac{\theta_2-\theta_1}{3}\right|^{-2k^2} +
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\pi/2+\epsilon/2}^{3\pi/2-\epsilon/2} d\theta_1 \ \mathcal{F}(\theta_1-\pi)^{k^2-1}\mathcal{F}(\theta_2)^{k^2-1}\left|3\sin\frac{\theta_2-\theta_1}{3}\right|^{-2k^2} \\
&= \frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-3\pi/2+\epsilon/2}^{-\pi/2-\epsilon/2} d\theta_1 \ \left|3\sin\frac{\theta_2-\theta_1}{3}\right|^{-2k^2} \times \\
&\qquad \times\left\{1+(k^2-1)\ln\mathcal{F}(\theta_1+\pi)+(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\}+ \\
&+\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\pi/2+\epsilon/2}^{3\pi/2-\epsilon/2} d\theta_1 \ \left|3\sin\frac{\theta_2-\theta_1}{3}\right|^{-2k^2} \times \\
&\qquad \times\left\{1+(k^2-1)\ln\mathcal{F}(\theta_1-\pi)+(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\} \\
&= \frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-3\pi/2+\epsilon/2}^{-\pi/2-\epsilon/2} d\theta_1 \ \left|3\sin\frac{\theta_2-\theta_1}{3}\right|^{-2k^2} \times \\
&\qquad \times\left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\}+ \\
&+\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\pi/2+\epsilon/2}^{3\pi/2-\epsilon/2} d\theta_1 \ \left|3\sin\frac{\theta_2-\theta_1}{3}\right|^{-2k^2} \times \\
&\qquad \times\left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\}. \qquad \text{(B.5)}
\end{aligned}
$$

It is convenient to define

$$
\begin{aligned}
K_{11/110}(n,k^2) \equiv\ & \frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-n\pi+\alpha(n)}^{-\pi/2-\epsilon/2} d\theta_1 \ \left|2n\sin\frac{\theta_2-\theta_1}{2n}\right|^{-2k^2} \times \\
&\times\left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\}+ \\
&+\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\pi/2+\epsilon/2}^{n\pi-\alpha(n)} d\theta_1 \ \left|2n\sin\frac{\theta_2-\theta_1}{2n}\right|^{-2k^2} \times \\
&\times\left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\}, \qquad \text{(B.6)}
\end{aligned}
$$

where $\alpha(n)=(n-1)\epsilon$. Since $K_{11/110}(n,k^2)$ is related to $K_{11}(k^2)$ and $K_{110}(k^2)$ by

$$
K_{11/110}(1,k^2)=\frac{1}{2}K_{11}(k^2)\,, \qquad K_{11/110}\left(\frac{3}{2},k^2\right)=K_{110}(k^2)\,, \qquad \text{(B.7)}
$$

$K(k^2)$ is given by

$$
\frac{1}{2}K(k^2)=K_{11/110}(1,k^2)+K_{20/200}(1,k^2)-K_{11/110}\left(\frac{3}{2},k^2\right)-K_{20/200}\left(\frac{3}{2},k^2\right). \qquad \text{(B.8)}
$$

Note that n-independent terms in $K_{11/110}(n,k^2)$ and $K_{20/200}(n,k^2)$ do not contribute to $K(k^2)/2$.

As we did in subsection 4.2, $K_{11/110}(n,k^2)+K_{20/200}(n,k^2)$ can be rewritten in the following factorized form if we could neglect the divergence and set $\epsilon=0$:

$$
\begin{aligned}
K_{11/110}(n,k^2)\Big|_{\epsilon=0}+K_{20/200}(n,k^2)\Big|_{\epsilon=0} &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} d\theta_2 \int_{-n\pi}^{n\pi} d\theta_1 \left|2n\sin\frac{\theta_2-\theta_1}{2n}\right|^{-2k^2} \times \\
&\quad \times\left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\} \\
&= \frac{1}{2}\int_{-n\pi}^{n\pi} d\theta \left|2n\sin\frac{\theta}{2n}\right|^{-2k^2} \int_{-\pi/2}^{\pi/2} d\theta' \times \qquad \text{(B.9)} \\
&\quad \times\left\{1+2(k^2-1)\ln\mathcal{F}(\theta')\right\}+O\left((k^2-1)^2\right).
\end{aligned}
$$

Let us go back to the real case with a finite ϵ and try to bring the region of the integrals in $K_{11/110}(n,k^2)+K_{20/200}(n,k^2)$ to a form which is close to that of (B.9). One such form is given by

$$\begin{aligned}
&\int_{-\pi/2+3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi/2+\epsilon/2}^{\theta_2-\epsilon} d\theta_1 + \int_{-\pi/2+\epsilon/2}^{\pi/2-3\epsilon/2} d\theta_2 \int_{\theta_2+\epsilon}^{\pi/2-\epsilon/2} d\theta_1 + \\
&+\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-n\pi+\alpha(n)}^{-\pi/2-\epsilon/2} d\theta_1 + \int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\pi/2+\epsilon/2}^{n\pi-\alpha(n)} d\theta_1 = \\
&\quad = \int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \left\{ \int_{-n\pi}^{\theta_2-\epsilon} d\theta_1 + \int_{\theta_2+\epsilon}^{n\pi} d\theta_1 \right\} - \\
&\quad - \int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \left\{ \int_{-n\pi}^{-n\pi+\alpha(n)} d\theta_1 + \int_{n\pi-\alpha(n)}^{n\pi} d\theta_1 \right\} - \\
&\quad - \int_{-\pi/2+\epsilon/2}^{\pi/2-3\epsilon/2} d\theta_2 \int_{\pi/2-\epsilon/2}^{\pi/2+\epsilon/2} d\theta_1 - \int_{\pi/2-3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\theta_2+\epsilon}^{\pi/2+\epsilon/2} d\theta_1 - \\
&\quad - \int_{-\pi/2+3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi/2-\epsilon/2}^{-\pi/2+\epsilon/2} d\theta_1 - \int_{-\pi/2+\epsilon/2}^{-\pi/2+3\epsilon/2} d\theta_2 \int_{-\pi/2-\epsilon/2}^{\theta_2-\epsilon} d\theta_1 \,. \qquad \text{(B.10)}
\end{aligned}$$

The first term on the right-hand side of (B.10) corresponds to the region of the integrals in (B.9) and the remaining regions vanish in the limit $\epsilon \to 0$.

We divide $K_{11/110}(n,k^2)+K_{20/200}(n,k^2)$ into six parts $K_a(n,k^2)$, $K_b(n,k^2)$, $K_c(n,k^2)$, $K_d(n,k^2)$, $K_e(n,k^2)$, and $K_f(n,k^2)$ according to the six terms in (B.10):

$$\begin{aligned}
K_{11/110}(n,k^2)+K_{20/200}(n,k^2) &= K_a(n,k^2)+K_b(n,k^2)+K_c(n,k^2)+ \\
&\quad +K_d(n,k^2)+K_e(n,k^2)+K_f(n,k^2)\,. \qquad \text{(B.11)}
\end{aligned}$$

For example, $K_a(n,k^2)$ is defined by

$$\begin{aligned}
K_a(n,k^2) &= \frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-n\pi}^{\theta_2-\epsilon} d\theta_1 \left| 2n \sin\frac{\theta_2-\theta_1}{2n} \right|^{-2k^2} \times \\
&\quad \times \left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\} + \\
&\quad +\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\theta_2+\epsilon}^{n\pi} d\theta_1 \left| 2n \sin\frac{\theta_2-\theta_1}{2n} \right|^{-2k^2} \times \\
&\quad \times \left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\}. \qquad \text{(B.12)}
\end{aligned}$$

In terms of these six terms, $K(k^2)/2$ is given by

$$\begin{aligned}
\frac{K(k^2)}{2} &= \left[K_a(n,k^2)+K_b(n,k^2)+K_c(n,k^2)+K_d(n,k^2)+K_e(n,k^2)+K_f(n,k^2)\right]_{n=1} - \\
&\quad - \left[K_a(n,k^2)+K_b(n,k^2)+K_c(n,k^2)+K_d(n,k^2)+K_e(n,k^2)+K_f(n,k^2)\right]_{n=\frac{3}{2}}\,. \qquad \text{(B.13)}
\end{aligned}$$

Let us calculate each of the six terms on the right-hand side of (B.11).

$K_a(n, k^2)$. This is most important and factorizes as in the case of (B.9):

$$\begin{aligned} K_a(n,k^2) &= \frac{1}{2}\int_\epsilon^{2n\pi-\epsilon} d\theta \, \left|2n\sin\frac{\theta}{2n}\right|^{-2k^2} \times \\ &\times \int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta' \, \left\{1+2(k^2-1)\ln\mathcal{F}(\theta')\right\} + O\left((k^2-1)^2\right). \end{aligned} \tag{B.14}$$

The integral over θ reduces to the incomplete beta function by the change of variables:

$$y = \sin^2\frac{\theta}{2n}. \tag{B.15}$$

The result is

$$\begin{aligned} \int_\epsilon^{2n\pi-\epsilon} d\theta \left|2n\sin\frac{\theta}{2n}\right|^{-2k^2} &= (2n)^{-2k^2+1}\int_{\sin^2\frac{\epsilon}{2n}}^1 dy\, y^{-\frac{1}{2}-k^2}(1-y)^{-\frac{1}{2}} \\ &= (2n)^{-2k^2+1}\left[B\left(\frac{1}{2}-k^2,\frac{1}{2}\right) - B_{\sin^2\frac{\epsilon}{2n}}\left(\frac{1}{2}-k^2,\frac{1}{2}\right)\right], \end{aligned} \tag{B.16}$$

where $B_z(p,q)$ is defined by and expressed in terms of the hypergeometric function ${}_2F_1$ by

$$B_z(p,q) = \int_0^z dt\ t^{p-1}(1-t)^{q-1} = \frac{z^p}{p}\ {}_2F_1(p,1-q;p+1;z) \tag{B.17}$$

for $0 < \mathrm{Re}(z) < 1$. What is crucial in (B.16) is that its divergent part when $k^2 \simeq 1$ is independent of n:

$$(2n)^{-2k^2+1}B_{\sin^2\frac{\epsilon}{2n}}\left(\frac{1}{2}-k^2,\frac{1}{2}\right) = \frac{2}{1-2k^2}\epsilon^{1-2k^2}\left[1+O(\epsilon^2)\right]. \tag{B.18}$$

This is easily verified using the expression of $B_z(p,q)$ in terms of the hypergeometric function. Therefore, unless the integral over θ' becomes too singular in the limit $\epsilon \to 0$, the divergent part of $K_a(n,k^2)$ cancels in $K_a(1,k^2) - K_a(3/2,k^2)$, and we find the following finite contribution:

$$\begin{aligned} K_a(1,k^2) - K_a\left(\frac{3}{2},k^2\right) &= \frac{1}{2}(2^{-2k^2+1}-3^{-2k^2+1})B\left(\frac{1}{2}-k^2,\frac{1}{2}\right)\times \\ &\times\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta'\left\{1+2(k^2-1)\ln\mathcal{F}(\theta')\right\} + O\left((k^2-1)^2\right) + o(\epsilon), \end{aligned} \tag{B.19}$$

where $o(\epsilon)$ denotes terms which vanish in the limit $\epsilon \to 0$ as we defined in subsection 4.3. Since

$$B\left(\frac{1}{2}-k^2,\frac{1}{2}\right) = 2\pi(k^2-1) + O\left((k^2-1)^2\right), \tag{B.20}$$

we have

$$K_a(1,k^2) - K_a\left(\frac{3}{2},k^2\right) = \frac{\pi^2}{6}(k^2-1) + O\left((k^2-1)^2\right) + o(\epsilon). \tag{B.21}$$

$K_b(n,k^2)$. Since $K_b(1,k^2)$ vanishes because of $\alpha(1)=0$, consider $K_b(3/2,k^2)$ given by

$$\begin{aligned} K_b\left(\frac{3}{2},k^2\right) &= -\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-3\pi/2}^{-3\pi/2+\epsilon/2} d\theta_1 \left|3\sin\frac{\theta_2-\theta_1}{3}\right|^{-2k^2} \times \\ &\quad \times \left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\} - \\ &\quad -\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{3\pi/2-\epsilon/2}^{3\pi/2} d\theta_1 \left|3\sin\frac{\theta_2-\theta_1}{3}\right|^{-2k^2} \times \\ &\quad \times \left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\}. \end{aligned} \tag{B.22}$$

Since

$$\frac{\pi}{3} \le \left|\frac{\theta_2-\theta_1}{3}\right| < \frac{2\pi}{3}, \tag{B.23}$$

there is no singularity coming from the propagator. The integral over θ_1 is of order ϵ so that $K_b(3/2,k^2)$ vanishes unless a compensating factor emerges from the integral over θ_2.

$K_c(n,k^2)$ and $K_d(n,k^2)$. Let us first consider $K_c(n,k^2)$ which is defined by

$$\begin{aligned} K_c(n,k^2) &= -\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-3\epsilon/2} d\theta_2 \int_{\pi/2-\epsilon/2}^{\pi/2+\epsilon/2} d\theta_1 \left|2n\sin\frac{\theta_2-\theta_1}{2n}\right|^{-2k^2} \times \\ &\quad \times \left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\}. \end{aligned} \tag{B.24}$$

The integral over θ_1 can be written in terms of the incomplete beta function by the change of variables (B.15):

$$\begin{aligned} K_c(n,k^2) &= -\frac{1}{4}(2n)^{-2k^2+1}\int_{-\pi/2+\epsilon/2}^{\pi/2-3\epsilon/2} d\theta_2 \left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)\right\} \times \\ &\quad \times \left[B_{\sin^2\left(\frac{\pi}{4n}+\frac{\epsilon}{4n}-\frac{\theta_2}{2n}\right)}\left(\frac{1}{2}-k^2,\frac{1}{2}\right) - B_{\sin^2\left(\frac{\pi}{4n}-\frac{\epsilon}{4n}-\frac{\theta_2}{2n}\right)}\left(\frac{1}{2}-k^2,\frac{1}{2}\right)\right] + \\ &\quad + O\left((k^2-1)^2\right) \\ &= -\frac{1}{4}(2n)^{-2k^2+1}\int_{-\pi/2+\epsilon/2}^{\pi/2-3\epsilon/2} d\theta_2 \left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)\right\} \times \\ &\quad \times B_{\sin^2\left(\frac{\pi}{4n}+\frac{\epsilon}{4n}-\frac{\theta_2}{2n}\right)}\left(\frac{1}{2}-k^2,\frac{1}{2}\right) + \\ &\quad +\frac{1}{4}(2n)^{-2k^2+1}\int_{-\pi/2+3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2-\epsilon)\right\} \times \\ &\quad \times B_{\sin^2\left(\frac{\pi}{4n}+\frac{\epsilon}{4n}-\frac{\theta_2}{2n}\right)}\left(\frac{1}{2}-k^2,\frac{1}{2}\right) + O\left((k^2-1)^2\right). \end{aligned} \tag{B.25}$$

The calculation of $K_d(n,k^2)$,

$$\begin{aligned} K_d(n,k^2) &= -\frac{1}{2}\int_{\pi/2-3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\theta_2+\epsilon}^{\pi/2+\epsilon/2} d\theta_1 \left|2n\sin\frac{\theta_2-\theta_1}{2n}\right|^{-2k^2} \times \\ &\quad \times \left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\}, \end{aligned} \tag{B.26}$$

is almost the same as that of $K_c(n,k^2)$. The result is

$$\begin{aligned}
K_d(n,k^2) &= -\frac{1}{4}(2n)^{-2k^2+1}\int_{\pi/2-3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)\right\}\times \\
&\qquad \times\left[B_{\sin^2\left(\frac{\pi}{4n}+\frac{\epsilon}{4n}-\frac{\theta_2}{2n}\right)}\left(\frac{1}{2}-k^2,\frac{1}{2}\right)-B_{\sin^2\frac{\epsilon}{2n}}\left(\frac{1}{2}-k^2,\frac{1}{2}\right)\right]+ \\
&\quad +O\left((k^2-1)^2\right) \\
&= -\frac{1}{4}(2n)^{-2k^2+1}\int_{\pi/2-3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)\right\}\times \\
&\qquad \times B_{\sin^2\left(\frac{\pi}{4n}+\frac{\epsilon}{4n}-\frac{\theta_2}{2n}\right)}\left(\frac{1}{2}-k^2,\frac{1}{2}\right) \\
&\quad +\frac{1}{4}(2n)^{-2k^2+1}B_{\sin^2\frac{\epsilon}{2n}}\left(\frac{1}{2}-k^2,\frac{1}{2}\right)\int_{\pi/2-3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)\right\}+ \\
&\quad +O\left((k^2-1)^2\right). \qquad \text{(B.27)}
\end{aligned}$$

The sum of $K_c(n,k^2)$ and $K_d(n,k^2)$ is given by

$$\begin{aligned}
K_c(n,k^2)+K_d(n,k^2) &= -\frac{1}{4}(2n)^{-2k^2+1}\int_{\pi/2-3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \left\{1+2(k^2-1)\ln\mathcal{F}(-\theta_2)\right\}\times \\
&\qquad \times B_{\sin^2\left(\frac{\theta_2}{2n}+\frac{\pi}{4n}+\frac{\epsilon}{4n}\right)}\left(\frac{1}{2}-k^2,\frac{1}{2}\right)- \\
&\quad -\frac{1}{4}(2n)^{-2k^2+1}\int_{-\pi/2+\epsilon/2}^{\pi/2-3\epsilon/2} d\theta_2 \left\{2(k^2-1)\ln\frac{\mathcal{F}(-\theta_2)}{\mathcal{F}(-\theta_2-\epsilon)}\right\} \\
&\qquad \times B_{\sin^2\left(\frac{\theta_2}{2n}+\frac{\pi}{4n}+\frac{\epsilon}{4n}\right)}\left(\frac{1}{2}-k^2,\frac{1}{2}\right)+ \\
&\quad +\frac{1}{4}(2n)^{-2k^2+1}B_{\sin^2\frac{\epsilon}{2n}}\left(\frac{1}{2}-k^2,\frac{1}{2}\right)\times \qquad \text{(B.28)} \\
&\quad \times\int_{\pi/2-3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)\right\}+O\left((k^2-1)^2\right),
\end{aligned}$$

where we redefined $-\theta_2$ as θ_2 in the first two terms on the right-hand side for later convenience.

$K_e(n,k^2)$ and $K_f(n,k^2)$. The calculations of $K_e(n,k^2)$ and $K_f(n,k^2)$ defined by

$$\begin{aligned}
K_e(n,k^2) &= -\frac{1}{2}\int_{-\pi/2+3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi/2-\epsilon/2}^{-\pi/2+\epsilon/2} d\theta_1 \left|2n\sin\frac{\theta_2-\theta_1}{2n}\right|^{-2k^2}\times \\
&\qquad \times\left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\}, \qquad \text{(B.29)}
\end{aligned}$$

and

$$\begin{aligned}
K_f(n,k^2) &= -\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{-\pi/2+3\epsilon/2} d\theta_2 \int_{-\pi/2-\epsilon/2}^{\theta_2-\epsilon} d\theta_1 \left|2n\sin\frac{\theta_2-\theta_1}{2n}\right|^{-2k^2}\times \\
&\qquad \times\left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)+O\left((k^2-1)^2\right)\right\}, \qquad \text{(B.30)}
\end{aligned}$$

respectively, are completely parallel to those of $K_c(n,k^2)$ and $K_d(n,k^2)$. The sum of $K_e(n,k^2)$ and $K_f(n,k^2)$ is given by

$$\begin{aligned}
K_e(n,k^2)+K_f(n,k^2) \;=\; & -\frac{1}{4}(2n)^{-2k^2+1}\int_{\pi/2-3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)\right\}\times \\
& \qquad\qquad \times B_{\sin^2\left(\frac{\theta_2}{2n}+\frac{\pi}{4n}+\frac{\epsilon}{4n}\right)}\left(\frac{1}{2}-k^2,\frac{1}{2}\right)- \\
& -\frac{1}{4}(2n)^{-2k^2+1}\int_{-\pi/2+\epsilon/2}^{\pi/2-3\epsilon/2} d\theta_2 \left\{2(k^2-1)\ln\frac{\mathcal{F}(\theta_2)}{\mathcal{F}(\theta_2+\epsilon)}\right\}\times \\
& \qquad\qquad \times B_{\sin^2\left(\frac{\theta_2}{2n}+\frac{\pi}{4n}+\frac{\epsilon}{4n}\right)}\left(\frac{1}{2}-k^2,\frac{1}{2}\right)+ \\
& +\frac{1}{4}(2n)^{-2k^2+1}B_{\sin^2\frac{\epsilon}{2n}}\left(\frac{1}{2}-k^2,\frac{1}{2}\right)\times \\
& \times\int_{-\pi/2+\epsilon/2}^{-\pi/2+3\epsilon/2} d\theta_2 \left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)\right\}+O\left((k^2-1)^2\right).
\end{aligned} \tag{B.31}$$

Let us summarize the results. If $\ln\mathcal{F}(\theta)$ is not too singular in the limit $\epsilon\to 0$, $K_a(1,k^2)-K_a(3/2,k^2)$ is given by (B.21) and $K_b(n,k^2)$ vanishes. The remaining four terms are combined to give

$$\begin{aligned}
& K_c(n,k^2)+K_d(n,k^2)+K_e(n,k^2)+K_f(n,k^2)= \\
&\quad = -\frac{1}{2}(2n)^{-2k^2+1}\int_{\pi/2-3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2\, B_{\sin^2\left(\frac{\theta_2}{2n}+\frac{\pi}{4n}+\frac{\epsilon}{4n}\right)}\left(\frac{1}{2}-k^2,\frac{1}{2}\right)\times \\
&\qquad\qquad \times\left\{1+(k^2-1)\ln\left[\mathcal{F}(\theta_2)\mathcal{F}(-\theta_2)\right]\right\}- \\
&\quad\quad -\frac{1}{2}(2n)^{-2k^2+1}\int_{-\pi/2+\epsilon/2}^{\pi/2-3\epsilon/2} d\theta_2\, B_{\sin^2\left(\frac{\theta_2}{2n}+\frac{\pi}{4n}+\frac{\epsilon}{4n}\right)}\left(\frac{1}{2}-k^2,\frac{1}{2}\right)\times \\
&\qquad\qquad \times(k^2-1)\ln\left[\frac{\mathcal{F}(\theta_2)}{\mathcal{F}(\theta_2+\epsilon)}\frac{\mathcal{F}(-\theta_2)}{\mathcal{F}(-\theta_2-\epsilon)}\right]+ \\
&\quad\quad +\frac{1}{4}(2n)^{-2k^2+1}B_{\sin^2\frac{\epsilon}{2n}}\left(\frac{1}{2}-k^2,\frac{1}{2}\right)\times \\
&\quad\quad \times\int_{-\pi/2+\epsilon/2}^{-\pi/2+3\epsilon/2} d\theta_2\left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)\right\}+ \\
&\quad\quad +\frac{1}{4}(2n)^{-2k^2+1}B_{\sin^2\frac{\epsilon}{2n}}\left(\frac{1}{2}-k^2,\frac{1}{2}\right)\times \\
&\quad\quad \times\int_{\pi/2-3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2\left\{1+2(k^2-1)\ln\mathcal{F}(\theta_2)\right\}+O\left((k^2-1)^2\right).
\end{aligned} \tag{B.32}$$

The incomplete beta function in the first term on the right-hand side does not become singular as $\theta_2\to\pi/2$ so that this term vanishes in the limit $\epsilon\to 0$ unless $\ln\mathcal{F}(\theta_2)$ becomes too singular as $\theta_2\to\pm\pi/2$. If $\ln\mathcal{F}(\theta_2)$ becomes too singular as $\theta_2\to\pm\pi/2$, however, point-splitting regularization we are using will not be appropriate in the first place.

The second term on the right-hand side of (B.32) also vanishes in the limit $\epsilon \to 0$ because

$$\ln\left[\frac{\mathcal{F}(\theta_2)}{\mathcal{F}(\theta_2+\epsilon)}\frac{\mathcal{F}(-\theta_2)}{\mathcal{F}(-\theta_2-\epsilon)}\right] = \left\{\frac{\mathcal{F}'(-\theta_2)}{\mathcal{F}(-\theta_2)} - \frac{\mathcal{F}'(\theta_2)}{\mathcal{F}(\theta_2)}\right\}\epsilon + O(\epsilon^2). \tag{B.33}$$

The incomplete beta function becomes singular as $\theta_2 \to -\pi/2$, but it is not sufficient to compensate the factor (B.33) unless $\ln\mathcal{F}(\theta_2)$ becomes too singular as $\theta_2 \to \pm\pi/2$.

As for the third and fourth terms on the right-hand side of (B.32), their leading terms in the limit $\epsilon \to 0$ are independent of n because of (B.18) so that they do not contribute to $K(k^2)/2$. It would be more difficult for the integrals over θ_2 to provide compensating singular contributions than the case of $K_a(n, k^2)$.

Therefore, only $K_a(1, k^2) - K_a(3/2, k^2)$ contributes to $K(k^2)/2$ in the limit $\epsilon \to 0$, and $K(k^2)$ is given by

$$\begin{aligned} K(k^2) &= 2K_a(1,k^2) - 2K_a\left(\frac{3}{2},k^2\right) + o(\epsilon) \\ &= \frac{\pi^2}{3}(k^2-1) + O\left((k^2-1)^2\right) + o(\epsilon)\,, \end{aligned} \tag{B.34}$$

when $\ln\mathcal{F}(\theta)$ is not too singular.

Since the ϵ-dependence in $\mathcal{F}(\theta)$ depends on the choice of the Riemann surface Σ_n where the off-shell tachyon is defined, it would be difficult to evaluate possible singularity of $\ln\mathcal{F}(\theta)$ in general. We can easily verify, however, that the $\ln\epsilon$ singularity in $\ln\mathcal{F}(\theta)$ coming from the conformal transformation $z^{1/(n-1)}$,

$$\ln\mathcal{F}(\theta) = \ln\epsilon + \ln\frac{\mathcal{F}_0(\epsilon_0\theta/\epsilon)}{\epsilon_0}\,, \tag{B.35}$$

which follows from (4.15), does not change the result.

C. Another derivation of (4.28)

We found in subsection 4.1 that the kinetic terms of the open string fields $\{\varphi_i\}$ vanish when the fields satisfy the physical state conditions by explicit calculations. In this appendix, we show this using the conformal property of physical vertex operators.

For any pair of $\mathcal{O}$ and $\mathcal{O}'$ which are primary with conformal dimension one, let us define $\widetilde{K}$ by

$$\frac{1}{2}\widetilde{K} = \frac{1}{2}\widetilde{K}_{11} + \widetilde{K}_{20} - \widetilde{K}_{110} - \widetilde{K}_{200}\,, \tag{C.1}$$

where

$$\begin{aligned} \frac{1}{2}\widetilde{K}_{11} &= \frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi}^{-\pi/2-\epsilon/2} d\theta_1 \left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{\text{disk}} + \\ &\quad + \frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\pi/2+\epsilon/2}^{\pi} d\theta_1 \left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{\text{disk}}\,, \end{aligned} \tag{C.2}$$

$$\widetilde{K}_{20} = \frac{1}{2}\int_{-\pi/2+3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi/2+\epsilon/2}^{\theta_2-\epsilon} d\theta_1 \left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{\text{disk}} +$$

$$+\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-3\epsilon/2} d\theta_2 \int_{\theta_2+\epsilon}^{\pi/2-\epsilon/2} d\theta_1 \left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{\text{disk}}, \tag{C.3}$$

$$\begin{aligned}\widetilde{K}_{110} &= \frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-3\pi/2+\epsilon/2}^{-\pi/2-\epsilon/2} d\theta_1 \left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{3\pi} + \\ &+\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\pi/2+\epsilon/2}^{3\pi/2-\epsilon/2} d\theta_1 \left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{3\pi},\end{aligned} \tag{C.4}$$

$$\begin{aligned}\widetilde{K}_{200} &= \frac{1}{2}\int_{-\pi/2+3\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi/2+\epsilon/2}^{\theta_2-\epsilon} d\theta_1 \left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{3\pi} + \\ &+\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-3\epsilon/2} d\theta_2 \int_{\theta_2+\epsilon}^{\pi/2-\epsilon/2} d\theta_1 \left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{3\pi}.\end{aligned} \tag{C.5}$$

These definitions are obvious generalizations of $K(1)$, $K_{11}(1)$, $K_{20}(1)$, $K_{110}(1)$, and $K_{200}(1)$ we studied in subsection 4.1 and appendix B. Using (B.10) and the argument in appendix B showing that $K_b(n,k^2)$, $K_c(n,k^2)$, $K_d(n,k^2)$, $K_e(n,k^2)$, and $K_f(n,k^2)$ vanish in the limit $\epsilon \to 0$, we can show that $\widetilde{K}_{11}/2+\widetilde{K}_{20}$ can be written as

$$\begin{aligned}\frac{1}{2}\widetilde{K}_{11}+\widetilde{K}_{20} &= \frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-\pi}^{\theta_2-\epsilon} d\theta_1 \left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{\text{disk}} + \\ &+\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\theta_2+\epsilon}^{\pi} d\theta_1 \left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{\text{disk}} + o(\epsilon) \\ &= \frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\theta_2+\epsilon}^{\theta_2+2\pi-\epsilon} d\theta_1 \left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{\text{disk}} + o(\epsilon) \\ &= \frac{\pi-\epsilon}{2}\int_{\epsilon}^{2\pi-\epsilon} d\theta \left\langle \mathcal{O}(e^{i\theta})\mathcal{O}'(1)\right\rangle_{\text{disk}} + o(\epsilon),\end{aligned} \tag{C.6}$$

where we used

$$\left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{\text{disk}} = \left\langle \mathcal{O}(e^{i(\theta_1-\theta_2)})\mathcal{O}'(1)\right\rangle_{\text{disk}}. \tag{C.7}$$

Similarly for $\widetilde{K}_{110}+\widetilde{K}_{200}$, we have

$$\begin{aligned}\widetilde{K}_{110}+\widetilde{K}_{200} &= \frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{-3\pi/2}^{\theta_2-\epsilon} d\theta_1 \left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{3\pi} + \\ &+\frac{1}{2}\int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} d\theta_2 \int_{\theta_2+\epsilon}^{3\pi/2} d\theta_1 \left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{3\pi} + o(\epsilon).\end{aligned} \tag{C.8}$$

Let us make the conformal transformation $z^{2/3}$ for $\widetilde{K}_{110}+\widetilde{K}_{200}$. Since $\mathcal{O}$ and $\mathcal{O}'$ are conformal primary with dimension one, $\widetilde{K}_{110}+\widetilde{K}_{200}$ is transformed as

$$\begin{aligned}\widetilde{K}_{110}+\widetilde{K}_{200} &= \frac{1}{2}\int_{-\pi/3+\epsilon/3}^{\pi/3-\epsilon/3} d\theta_2 \int_{-\pi}^{\theta_2-2\epsilon/3} d\theta_1 \left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{\text{disk}} + \\ &+\frac{1}{2}\int_{-\pi/3+\epsilon/3}^{\pi/3-\epsilon/3} d\theta_2 \int_{\theta_2+2\epsilon/3}^{\pi} d\theta_1 \left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{\text{disk}} + o(\epsilon) \\ &= \frac{1}{2}\int_{-\pi/3+\epsilon/3}^{\pi/3-\epsilon/3} d\theta_2 \int_{\theta_2+2\epsilon/3}^{\theta_2+2\pi-2\epsilon/3} d\theta_1 \left\langle \mathcal{O}(e^{i\theta_1})\mathcal{O}'(e^{i\theta_2})\right\rangle_{\text{disk}} + o(\epsilon)\end{aligned}$$

$$= \frac{\pi - \epsilon}{3} \int_{2\epsilon/3}^{2\pi - 2\epsilon/3} d\theta \left\langle \mathcal{O}(e^{i\theta}) \mathcal{O}'(1) \right\rangle_{\text{disk}} + o(\epsilon) \,. \tag{C.9}$$

Let us make a further conformal transformation to (C.9) such that the integral is from ϵ to $2\pi - \epsilon$. Consider a class of conformal transformations parametrized by a real, positive constant a,

$$f(z) = \frac{(1+a)z + 1 - a}{(1-a)z + 1 + a} \,, \qquad \left. \frac{df(z)}{dz} \right|_{z=1} = a \,, \tag{C.10}$$

which maps the unit disk to itself:

$$|f(e^{i\theta})| = 1 \,, \qquad |f(0)| < 1 \,. \tag{C.11}$$

The constant a is determined as

$$a = \frac{\tan \frac{\epsilon}{2}}{\tan \frac{\epsilon}{3}} \tag{C.12}$$

by the condition that

$$f(e^{\pm 2i\epsilon/3}) = e^{\pm i\epsilon} \,. \tag{C.13}$$

While the integral of $\mathcal{O}(e^{i\theta})$ over θ remains invariant, the operator $\mathcal{O}'(1)$ is transformed as

$$\mathcal{O}'(1) \to \frac{\tan \frac{\epsilon}{2}}{\tan \frac{\epsilon}{3}} \, \mathcal{O}'(1) \,. \tag{C.14}$$

By this conformal transformation, the last line of (C.9) is transformed as

$$\widetilde{K}_{110} + \widetilde{K}_{200} = \frac{\pi - \epsilon}{3} \frac{\tan \frac{\epsilon}{2}}{\tan \frac{\epsilon}{3}} \int_{\epsilon}^{2\pi - \epsilon} d\theta \left\langle \mathcal{O}(e^{i\theta}) \mathcal{O}'(1) \right\rangle_{\text{disk}} + o(\epsilon) \,. \tag{C.15}$$

Therefore, $\widetilde{K}/2$ is given by

$$\frac{1}{2}\widetilde{K} = \frac{\pi - \epsilon}{2} \left(1 - \frac{2}{3} \frac{\tan \frac{\epsilon}{2}}{\tan \frac{\epsilon}{3}} \right) \int_{\epsilon}^{2\pi - \epsilon} d\theta \left\langle \mathcal{O}(e^{i\theta}) \mathcal{O}'(1) \right\rangle_{\text{disk}} + o(\epsilon) \,. \tag{C.16}$$

Since $\mathcal{O}$ and $\mathcal{O}'$ are primary with conformal dimension one, the singularity of the propagator is

$$\left\langle \mathcal{O}(e^{i\theta_1}) \mathcal{O}'(e^{i\theta_2}) \right\rangle_{\text{disk}} \sim \frac{1}{(\theta_1 - \theta_2)^2} \tag{C.17}$$

when $\theta_1 \sim \theta_2$ so that

$$\int_{\epsilon}^{2\pi - \epsilon} d\theta \left\langle \mathcal{O}(e^{i\theta}) \mathcal{O}'(1) \right\rangle_{\text{disk}} = O\left(\frac{1}{\epsilon} \right) . \tag{C.18}$$

On the other hand, the factor in front of the integral scales in the limit $\epsilon \to 0$ as

$$1 - \frac{2}{3} \frac{\tan \frac{\epsilon}{2}}{\tan \frac{\epsilon}{3}} = O(\epsilon^2) \,, \tag{C.19}$$

therefore

$$\frac{1}{2}\widetilde{K} = o(\epsilon) \,. \tag{C.20}$$

This explains that the cancellation of the finite terms between (4.20) and (4.27) in subsection 4.1 is not accidental but a consequence of the conformal property of the physical vertex operators.

D. $V_{30} + V_{21} - V_{300} - V_{210} - V_{201}$

We calculate V_{30}, V_{300}, V_{21}, V_{210}, and V_{201} defined in subsection 4.3 to show that $V_{30} + V_{21} - V_{300} - V_{210} - V_{201}$ vanishes in the limit $\epsilon \to 0$.

Definitions. We regularize the state $|\mathit{3}\rangle$ by regularizing the integrals of the inserted vertex operators in the cone representation as follows:

$$\int_{-(n-1)\pi/2+5\epsilon_0/2}^{(n-1)\pi/2-\epsilon_0/2} d\theta_3 \int_{-(n-1)\pi/2+3\epsilon_0/2}^{\theta_3-\epsilon_0} d\theta_2 \int_{-(n-1)\pi/2+\epsilon_0/2}^{\theta_2-\epsilon_0} d\theta_1 \times \\ \times \mathcal{F}_0(\theta_1)^{k^2-1} e^{ikX(e^{i\theta_1})} \mathcal{F}_0(\theta_2)^{k^2-1} e^{ikX(e^{i\theta_2})} \mathcal{F}_0(\theta_3)^{k^2-1} e^{ikX(e^{i\theta_3})}. \tag{D.1}$$

The propagators in V_{30} and V_{300} are given in a similar way as in the case of V_{111} in (4.46). It is convenient to define $V_{30/300}(n)$ by

$$V_{30/300}(n) \equiv \int_{5\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \int_{3\epsilon/2}^{\theta_3-\epsilon} d\theta_2 \int_{\epsilon/2}^{\theta_2-\epsilon} d\theta_1 \times \\ \times \left(2n \sin\frac{\theta_2-\theta_1}{2n}\right)^{-1} \left(2n \sin\frac{\theta_3-\theta_1}{2n}\right)^{-1} \left(2n \sin\frac{\theta_3-\theta_2}{2n}\right)^{-1}. \tag{D.2}$$

This is related to V_{30} and V_{300} as

$$V_{30} = V_{30/300}(1)\,, \qquad V_{300} = V_{30/300}\left(\frac{3}{2}\right). \tag{D.3}$$

As for V_{21}, V_{210}, and V_{201}, we define $V_{21/210+201}(n)$ by

$$V_{21/210+201}(n) \equiv \int_{C_n} d\theta_3 \int_{-\pi+3\epsilon/2}^{-\epsilon/2} d\theta_2 \int_{-\pi+\epsilon/2}^{\theta_2-\epsilon} d\theta_1 \times \\ \times \left(2n \sin\frac{\theta_2-\theta_1}{2n}\right)^{-1} \left(2n \sin\frac{\theta_3-\theta_1}{2n}\right)^{-1} \left(2n \sin\frac{\theta_3-\theta_2}{2n}\right)^{-1}, \tag{D.4}$$

where

$$\int_{C_1} d\theta_3 = \int_{\epsilon/2}^{\pi-\epsilon/2}, \qquad \int_{C_{3/2}} d\theta_3 = \int_{\epsilon/2}^{\pi-\epsilon/2} + \int_{\pi+\epsilon/2}^{2\pi-\epsilon/2}. \tag{D.5}$$

This is related to V_{21} and $V_{210} + V_{201}$ as

$$V_{21} = V_{21/210+201}(1)\,, \qquad V_{210} + V_{201} = V_{21/210+201}\left(\frac{3}{2}\right). \tag{D.6}$$

The contour C_n can also be written as

$$\int_{C_n} d\theta_3 = \int_{\epsilon/2}^{(2n-1)\pi-\epsilon/2} d\theta_3 + \int_{\widetilde{C}_n} d\theta_3\,, \tag{D.7}$$

where

$$\int_{\widetilde{C}_1} d\theta_3 = 0\,, \qquad \int_{\widetilde{C}_{3/2}} d\theta_3 = \int_{\pi+\epsilon/2}^{\pi-\epsilon/2} d\theta_3\,. \tag{D.8}$$

As we will see, integrals along $\widetilde{C}_n$ are unimportant in most cases.

Using $V_{30/300}(n)$ and $V_{21/210+201}(n)$, $V_{30}+V_{21}-V_{300}-V_{210}-V_{201}$ is expressed as

$$V_{30}+V_{21}-V_{300}-V_{210}-V_{201} = V_{30/300}(1)+V_{21/210+201}(1)-V_{30/300}\left(\frac{3}{2}\right)- \\ -V_{21/210+201}\left(\frac{3}{2}\right). \tag{D.9}$$

Let us calculate $V_{30/300}(n)$ and $V_{21/210+201}(n)$.

$\boldsymbol{V_{30/300}(n)}$**.** The calculations of the integrals over θ_1 and θ_2 are tedious but straightforward:

$$\begin{aligned} V_{30/300}(n) &= \frac{1}{(2n)^2}\int_{5\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \int_{3\epsilon/2}^{\theta_3-\epsilon} d\theta_2 \left(\sin\frac{\theta_3-\theta_2}{2n}\right)^{-2} \ln\left(\frac{\sin\frac{\theta_3-\theta_2+\epsilon}{2n}}{\sin\frac{\epsilon}{2n}}\frac{\sin\frac{\theta_2-\epsilon/2}{2n}}{\sin\frac{\theta_3-\epsilon/2}{2n}}\right) \\ &= \frac{1}{n}\int_{5\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \left\{-\frac{\theta_3-5\epsilon/2}{2n}+\cot\frac{\theta_3-\epsilon/2}{2n}\ln\frac{\sin\frac{\epsilon}{2n}}{\sin\frac{\theta_3-3\epsilon/2}{2n}}+\right. \\ &\qquad\left. +\cot\frac{\epsilon}{2n}\ln\left(\frac{\sin\frac{\epsilon}{n}}{\sin\frac{\epsilon}{2n}}\frac{\sin\frac{\theta_3-3\epsilon/2}{2n}}{\sin\frac{\theta_3-\epsilon/2}{2n}}\right)\right\}. \end{aligned} \tag{D.10}$$

We divide this expression into the following six terms:

$$\begin{aligned} V_{30/300}(n) &= -\frac{1}{n}\int_{5\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \frac{\theta_3-5\epsilon/2}{2n}+\frac{1}{n}\int_{5\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \cot\frac{\theta_3-\epsilon/2}{2n}\ln\sin\frac{\epsilon}{2n}- \\ &\quad -\frac{1}{n}\int_{5\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \cot\frac{\theta_3-\epsilon/2}{2n}\ln\frac{\sin\frac{\theta_3-3\epsilon/2}{2n}}{\sin\frac{\theta_3-\epsilon/2}{2n}}- \\ &\quad -\frac{1}{n}\int_{5\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \cot\frac{\theta_3-\epsilon/2}{2n}\ln\sin\frac{\theta_3-\epsilon/2}{2n}+ \\ &\quad +\frac{1}{n}\int_{5\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \cot\frac{\epsilon}{2n}\ln\frac{\sin\frac{\epsilon}{n}}{\sin\frac{\epsilon}{2n}}+\frac{1}{n}\int_{5\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \cot\frac{\epsilon}{2n}\ln\frac{\sin\frac{\theta_3-3\epsilon/2}{2n}}{\sin\frac{\theta_3-\epsilon/2}{2n}}. \end{aligned} \tag{D.11}$$

The first, second, fourth, and fifth integrals are easily carried out to give

$$\begin{aligned} -\frac{1}{n}\int_{5\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \frac{\theta_3-5\epsilon/2}{2n} &= -\frac{\pi^2}{(2n)^2}+o(\epsilon)\,, \\ \frac{1}{n}\int_{5\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \cot\frac{\theta_3-\epsilon/2}{2n}\ln\sin\frac{\epsilon}{2n} &= 2\ln\sin\frac{\epsilon}{2n}\ln\frac{\sin\frac{\pi}{2n}}{\sin\frac{\epsilon}{n}}+o(\epsilon)\,, \\ -\frac{1}{n}\int_{5\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \cot\frac{\theta_3-\epsilon/2}{2n}\ln\sin\frac{\theta_3-\epsilon/2}{2n} &= \left(\ln\sin\frac{\epsilon}{n}\right)^2-\left(\ln\sin\frac{\pi}{2n}\right)^2+o(\epsilon)\,, \\ \frac{1}{n}\int_{5\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \cot\frac{\epsilon}{2n}\ln\frac{\sin\frac{\epsilon}{n}}{\sin\frac{\epsilon}{2n}} &= \frac{2\pi\ln 2}{\epsilon}-6\ln 2+o(\epsilon)\,. \end{aligned} \tag{D.12}$$

The third integral in (D.11) is finite and independent of n in the limit $\epsilon \to 0$. This can be seen by the following change of variables:

$$-\frac{1}{n}\int_{5\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \cot\frac{\theta_3-\epsilon/2}{2n}\ln\frac{\sin\frac{\theta_3-3\epsilon/2}{2n}}{\sin\frac{\theta_3-\epsilon/2}{2n}} = -2\int_{\frac{2n\epsilon}{\pi-\epsilon}}^{n} dx\frac{\epsilon}{x^2}\cot\frac{\epsilon}{x}\ln\frac{\sin\left(\frac{\epsilon}{x}-\frac{\epsilon}{2n}\right)}{\sin\frac{\epsilon}{x}}. \quad \text{(D.13)}$$

We can take the limit $\epsilon \to 0$ to find

$$\begin{aligned}
-2\int_0^n \frac{dx}{x}\ln\frac{\frac{1}{x}-\frac{1}{2n}}{\frac{1}{x}} + o(\epsilon) &= -2\int_0^1 \frac{dx}{x}\ln\left(1-\frac{x}{2}\right) + o(\epsilon)\\
&= 2\,\mathrm{Li}_2\left(\frac{1}{2}\right) + o(\epsilon) = \frac{\pi^2}{6} - (\ln 2)^2 + o(\epsilon)\,, \qquad \text{(D.14)}
\end{aligned}$$

where the polylogarithm $\mathrm{Li}_2(z)$ is defined by

$$\mathrm{Li}_2(z) = -\int_0^z dt\,\frac{\ln(1-t)}{t}\,. \qquad \text{(D.15)}$$

The finite value in (D.14) is not important because it is independent of n. Finally, we rewrite the last integral in (D.11) as follows:

$$\begin{aligned}
\frac{1}{n}\int_{5\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \cot\frac{\epsilon}{2n}\ln\frac{\sin\frac{\theta_3-3\epsilon/2}{2n}}{\sin\frac{\theta_3-\epsilon/2}{2n}} &= \frac{1}{n}\cot\frac{\epsilon}{2n}\left\{\int_\epsilon^{2\epsilon} d\theta\ln\sin\frac{\theta}{2n} - \int_{\pi-2\epsilon}^{\pi-\epsilon} d\theta\ln\sin\frac{\theta}{2n}\right\}\\
&= \frac{1}{n}\cot\frac{\epsilon}{2n}\int_\epsilon^{2\epsilon} d\theta\ln\sin\frac{\theta}{2n} - 2\ln\sin\frac{\pi}{2n} + o(\epsilon)\,. \quad \text{(D.16)}
\end{aligned}$$

The final result for $V_{30/300}(n)$ is given by

$$\begin{aligned}
V_{30/300}(n) &= -\frac{\pi^2}{(2n)^2} + 2\ln\sin\frac{\epsilon}{2n}\ln\frac{\sin\frac{\pi}{2n}}{\sin\frac{\epsilon}{n}} + \\
&\quad + \frac{\pi^2}{6} - (\ln 2)^2 + \left(\ln\sin\frac{\epsilon}{n}\right)^2 - \left(\ln\sin\frac{\pi}{2n}\right)^2 + \frac{2\pi\ln 2}{\epsilon} - 6\ln 2 + \\
&\quad + \frac{1}{n}\cot\frac{\epsilon}{2n}\int_\epsilon^{2\epsilon} d\theta\ln\sin\frac{\theta}{2n} - 2\ln\sin\frac{\pi}{2n} + o(\epsilon)\,. \qquad \text{(D.17)}
\end{aligned}$$

$\boldsymbol{V_{21/210+201}(n)}$**.** Let us move on to $V_{21/210+201}(n)$. The calculations of the integrals over θ_1 and θ_2 are again tedious but straightforward:

$$\begin{aligned}
V_{21/210+201}(n) &= \frac{1}{(2n)^2}\int_{C_n} d\theta_3 \int_{-\pi+3\epsilon/2}^{-\epsilon/2} d\theta_2 \left(\sin\frac{\theta_3-\theta_2}{2n}\right)^{-2}\ln\left(\frac{\sin\frac{\theta_3-\theta_2+\epsilon}{2n}}{\sin\frac{\epsilon}{2n}}\frac{\sin\frac{\theta_2+\pi-\epsilon/2}{2n}}{\sin\frac{\theta_3+\pi-\epsilon/2}{2n}}\right)\\
&= \frac{1}{n}\int_{C_n} d\theta_3 \Bigg\{ -\frac{\pi-2\epsilon}{2n} + \cot\frac{\theta_3+\epsilon/2}{2n}\ln\left(\frac{\sin\frac{\pi-\epsilon}{2n}}{\sin\frac{\epsilon}{2n}}\frac{\sin\frac{\theta_3+3\epsilon/2}{2n}}{\sin\frac{\theta_3+\pi-\epsilon/2}{2n}}\right) + \\
&\qquad + \cot\frac{\epsilon}{2n}\ln\frac{\sin\frac{\theta_3+3\epsilon/2}{2n}}{\sin\frac{\theta_3+\epsilon/2}{2n}}\Bigg\}. \qquad \text{(D.18)}
\end{aligned}$$

We divide this expression into the following six terms:

$$\begin{aligned}
V_{21/210+201}(n) &= -\frac{1}{n}\int_{C_n} d\theta_3 \frac{\pi-2\epsilon}{2n} + \frac{1}{n}\int_{C_n} d\theta_3 \cot\frac{\theta_3+\epsilon/2}{2n}\ln\frac{\sin\frac{\pi-\epsilon}{2n}}{\sin\frac{\epsilon}{2n}} + \\
&\quad +\frac{1}{n}\int_{C_n} d\theta_3 \cot\frac{\theta_3+\epsilon/2}{2n}\ln\frac{\sin\frac{\theta_3+3\epsilon/2}{2n}}{\sin\frac{\theta_3+\epsilon/2}{2n}} + \\
&\quad +\frac{1}{n}\int_{C_n} d\theta_3 \cot\frac{\theta_3+\epsilon/2}{2n}\ln\sin\frac{\theta_3+\epsilon/2}{2n} - \\
&\quad -\frac{1}{n}\int_{C_n} d\theta_3 \cot\frac{\theta_3+\epsilon/2}{2n}\ln\sin\frac{\theta_3+\pi-\epsilon/2}{2n} + \\
&\quad +\frac{1}{n}\int_{C_n} d\theta_3 \cot\frac{\epsilon}{2n}\ln\frac{\sin\frac{\theta_3+3\epsilon/2}{2n}}{\sin\frac{\theta_3+\epsilon/2}{2n}}\,. \qquad \text{(D.19)}
\end{aligned}$$

The first, second, and fourth integrals in (D.19) are easily carried out. The contributions from the contour $\widetilde{C}_n$ defined by (D.8) to these integrals vanish in the limit $\epsilon \to 0$ so that we can replace the contour C_n by the first term on the right-hand side of (D.7). The three integrals are carried out as follows:

$$\begin{aligned}
-\frac{1}{n}\int_{\epsilon/2}^{(2n-1)\pi-\epsilon/2} d\theta_3\frac{\pi-2\epsilon}{2n} &= -\frac{(2n-1)\pi^2}{2n^2} + o(\epsilon)\,, \\
\frac{1}{n}\int_{\epsilon/2}^{(2n-1)\pi-\epsilon/2} d\theta_3\cot\frac{\theta_3+\epsilon/2}{2n}\ln\frac{\sin\frac{\pi-\epsilon}{2n}}{\sin\frac{\epsilon}{2n}} &= 2\left(\ln\frac{\sin\frac{\pi}{2n}}{\sin\frac{\epsilon}{2n}}\right)^2 + o(\epsilon)\,, \\
\frac{1}{n}\int_{\epsilon/2}^{(2n-1)\pi-\epsilon/2} d\theta_3\cot\frac{\theta_3+\epsilon/2}{2n}\ln\sin\frac{\theta_3+\epsilon/2}{2n} &= \left(\ln\sin\frac{\pi}{2n}\right)^2 - \left(\ln\sin\frac{\epsilon}{2n}\right)^2 . \qquad \text{(D.20)}
\end{aligned}$$

The third integral in (D.19) is finite and independent of n. We can show this in a similar way as in the case of the third integral in (D.11). The contribution from the contour $\widetilde{C}_n$ vanishes in the limit $\epsilon \to 0$, and the remaining part gives

$$\begin{aligned}
\frac{1}{n}\int_{\epsilon/2}^{(2n-1)\pi-\epsilon/2} d\theta_3\cot\frac{\theta_3+\epsilon/2}{2n}\ln\frac{\sin\frac{\theta_3+3\epsilon/2}{2n}}{\sin\frac{\theta_3+\epsilon/2}{2n}} &= 2\int_{\frac{2n\epsilon}{(2n-1)\pi}}^{2n} dx\frac{\epsilon}{x^2}\cot\frac{\epsilon}{x}\ln\frac{\sin\left(\frac{\epsilon}{x}+\frac{\epsilon}{2n}\right)}{\sin\frac{\epsilon}{x}} \\
&= 2\int_0^2 \frac{dx}{x}\ln\left(1+\frac{x}{2}\right) + o(\epsilon) \\
&= 2\,\mathrm{Li}_2(-1) + o(\epsilon) = \frac{\pi^2}{6} + o(\epsilon)\,. \qquad \text{(D.21)}
\end{aligned}$$

The n-independent finite value here is again unimportant.

For the fifth integral in (D.19), we could not calculate it for a generic value of n. In the case of $n = 1$, the integral is finite in the limit $\epsilon \to 0$ and given by

$$\begin{aligned}
-\int_{\epsilon/2}^{\pi-\epsilon/2} d\theta_3\cot\frac{\theta_3+\epsilon/2}{2}\ln\sin\frac{\theta_3+\pi-\epsilon/2}{2} &= -\int_0^\pi d\theta_3\cot\frac{\theta_3}{2}\ln\cos\frac{\theta_3}{2} + o(\epsilon) \\
&= -2\int_0^1\frac{dx}{x}\ln\sqrt{1-x^2} + o(\epsilon)
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Li}_2(1) + \mathrm{Li}_2(-1) + o(\epsilon) \\
&= \frac{\pi^2}{12} + o(\epsilon)\,, \qquad \text{(D.22)}
\end{aligned}
$$

where we changed the variable as $x = \sin(\theta/2)$. For $n = 3/2$, we have

$$
\begin{aligned}
&-\frac{2}{3}\int_{\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \cot\frac{\theta_3+\epsilon/2}{3}\ln\sin\frac{\theta_3+\pi-\epsilon/2}{3} - \\
&-\frac{2}{3}\int_{\pi+\epsilon/2}^{2\pi-\epsilon/2} d\theta_3 \cot\frac{\theta_3+\epsilon/2}{3}\ln\sin\frac{\theta_3+\pi-\epsilon/2}{3} = \\
&\quad = -\left[2\ln\sin\frac{\theta_3+\epsilon/2}{3}\;\ln\sin\frac{\theta_3+\pi-\epsilon/2}{3}\right]_{\theta_3=\epsilon/2}^{\theta_3=\pi-\epsilon/2} + \\
&\qquad +\frac{2}{3}\int_{\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \ln\sin\frac{\theta_3+\epsilon/2}{3}\cot\frac{\theta_3+\pi-\epsilon/2}{3} - \\
&\qquad -\frac{2}{3}\int_{\pi+\epsilon/2}^{2\pi-\epsilon/2} d\theta_3 \cot\frac{\theta_3+\epsilon/2}{3}\ln\sin\frac{\theta_3+\pi-\epsilon/2}{3} \\
&\quad = -2\ln\sin\frac{\pi-\epsilon}{3}\ln\sin\frac{2\pi-\epsilon}{3} + 2\ln\sin\frac{\epsilon}{3}\ln\sin\frac{\pi}{3} + \\
&\qquad +\frac{4}{3}\int_{\epsilon/2}^{\pi-\epsilon/2} d\theta_3 \ln\sin\frac{\theta_3+\epsilon/2}{3}\cot\frac{\theta_3+\pi-\epsilon/2}{3} \\
&\quad = -2\left(\ln\sin\frac{\pi}{3}\right)^2 + 2\ln\sin\frac{\epsilon}{3}\ln\sin\frac{\pi}{3} + \frac{4}{3}\int_0^{\pi} d\theta\cot\frac{\theta+\pi}{3}\ln\sin\frac{\theta}{3} + o(\epsilon)\,. \qquad \text{(D.23)}
\end{aligned}
$$

The calculation of the integral in the last line is slightly complicated so that we postpone it and finish the remaining part of $V_{21/210+201}(n)$.

For the last integral in (D.19), we divide it into two parts according to (D.7). For the first contour, we have

$$
\begin{aligned}
\frac{1}{n}\int_{\epsilon/2}^{(2n-1)\pi-\epsilon/2} d\theta_3\;\cot\frac{\epsilon}{2n}\ln\frac{\sin\frac{\theta_3+3\epsilon/2}{2n}}{\sin\frac{\theta_3+\epsilon/2}{2n}} &= \frac{1}{n}\cot\frac{\epsilon}{2n}\left\{-\int_{\epsilon}^{2\epsilon} d\theta\;\ln\sin\frac{\theta}{2n} + \right. \\
&\qquad \left. + \int_{(2n-1)\pi}^{(2n-1)\pi+\epsilon} d\theta\;\ln\sin\frac{\theta}{2n}\right\} \qquad \text{(D.24)} \\
&= -\frac{1}{n}\cot\frac{\epsilon}{2n}\int_{\epsilon}^{2\epsilon} d\theta\;\ln\sin\frac{\theta}{2n} + 2\ln\sin\frac{\pi}{2n} + o(\epsilon)\,.
\end{aligned}
$$

The integral over $\widetilde{C}_n$ does vanish in the limit $\epsilon \to 0$, but in a slightly subtle way:

$$
\begin{aligned}
\frac{2}{3}\int_{\pi+\epsilon/2}^{\pi-\epsilon/2} d\theta_3\;\cot\frac{\epsilon}{3}\ln\frac{\sin\frac{\theta_3+3\epsilon/2}{3}}{\sin\frac{\theta_3+\epsilon/2}{3}} &= \frac{2}{3}\cot\frac{\epsilon}{3}\left\{\int_{\pi+2\epsilon}^{\pi+\epsilon} d\theta\;\ln\sin\frac{\theta}{3} - \int_{\pi+\epsilon}^{\pi} d\theta\;\ln\sin\frac{\theta}{3}\right\} \\
&= 2\left(\ln\sin\frac{\pi}{3} - \ln\sin\frac{\pi+\epsilon}{3}\right) + o(\epsilon) = o(\epsilon)\,. \qquad \text{(D.25)}
\end{aligned}
$$

The final result for $V_{21/210+201}(n)$ is given by

$$
\begin{aligned}
V_{21/210+201}(n) &= -\frac{(2n-1)\pi^2}{2n^2} + 2\left(\ln\frac{\sin\frac{\pi}{2n}}{\sin\frac{\epsilon}{2n}}\right)^2 + \frac{\pi^2}{6} - \\
&\quad - \left(\ln\sin\frac{\epsilon}{2n}\right)^2 - \left(\ln\sin\frac{\pi}{2n}\right)^2 + 2\ln\sin\frac{\epsilon}{2n}\ln\sin\frac{\pi}{2n} + I_n - \\
&\quad - \frac{1}{n}\cot\frac{\epsilon}{2n}\int_{\epsilon}^{2\epsilon} d\theta\,\ln\sin\frac{\theta}{2n} + 2\ln\sin\frac{\pi}{2n} + o(\epsilon),
\end{aligned} \tag{D.26}
$$

where

$$
I_1 = \frac{\pi^2}{12}, \qquad I_{3/2} = \frac{4}{3}\int_0^{\pi} d\theta\,\cot\frac{\theta+\pi}{3}\ln\sin\frac{\theta}{3}. \tag{D.27}
$$

Calculation of $I_{3/2}$. Let us calculate the integral $I_{3/2}$ in (D.27). It can be transformed to the following form by change of variables:

$$
\begin{aligned}
I_{3/2} &= -\frac{4}{3}\int_{-\pi/2}^{\pi/2} d\theta\,\tan\frac{\theta}{3}\ln\sin\left(\frac{\theta}{3}+\frac{\pi}{6}\right) \\
&= -\frac{2}{3}\int_{-\pi/2}^{\pi/2} d\theta\,\tan\frac{\theta}{3}\left[\ln\sin\left(\frac{\pi}{6}+\frac{\theta}{3}\right) - \ln\sin\left(\frac{\pi}{6}-\frac{\theta}{3}\right)\right] \\
&= -2\int_{-1/\sqrt{3}}^{1/\sqrt{3}} dy\,\frac{y}{1+y^2}\ln\frac{1+\sqrt{3}y}{1-\sqrt{3}y},
\end{aligned} \tag{D.28}
$$

where y in the last line is related to θ in the previous line by $y = \tan(\theta/3)$. The integral can be expressed in terms of polylogarithms. We find

$$
\begin{aligned}
I_{3/2} &= i\int_{-1/\sqrt{3}}^{1/\sqrt{3}} dy\,\left(\frac{1}{1-iy} - \frac{1}{1+iy}\right)\ln\frac{1+\sqrt{3}y}{1-\sqrt{3}y} \\
&= 2i\int_{-1/\sqrt{3}}^{1/\sqrt{3}} dy\,\frac{1}{1-iy}\ln\frac{1+\sqrt{3}y}{1-\sqrt{3}y} \\
&= -2\int_{i-1/\sqrt{3}}^{i+1/\sqrt{3}} dy\,\frac{1}{y}\left\{\ln\frac{1-\sqrt{3}i}{1+\sqrt{3}i} + \ln\left(1+\frac{\sqrt{3}y}{1-\sqrt{3}i}\right) - \ln\left(1-\frac{\sqrt{3}y}{1+\sqrt{3}i}\right)\right\} \\
&= \frac{4\pi^2}{9} - 2\int_1^{1/2-\sqrt{3}i/2} dy\,\frac{1}{y}\ln(1-y) + 2\int_{1/2+\sqrt{3}i/2}^{1} dy\,\frac{1}{y}\ln(1-y) \\
&= \frac{4\pi^2}{9} - 4\,\mathrm{Li}_2(1) + 2\,\mathrm{Li}_2(e^{\pi i/3}) + 2\,\mathrm{Li}_2(e^{-\pi i/3}) \\
&= -\frac{2\pi^2}{9} + 2\,\mathrm{Li}_2(e^{\pi i/3}) + 2\,\mathrm{Li}_2(e^{-\pi i/3}).
\end{aligned} \tag{D.29}
$$

Using the formula[18]

$$
\mathrm{Li}_2(e^{2\pi i p/q}) = \frac{1}{q^2}\sum_{k=1}^{q-1} e^{2\pi i k p/q}\psi^{(1)}\left(\frac{k}{q}\right) + \frac{\pi^2}{6q^2}, \tag{D.30}
$$

[18]This formula can be found, for example, at
`http://functions.wolfram.com/ZetaFunctionsandPolylogarithms/PolyLog2/03/01/`

where p and q $(0 < p \leq q)$ are integers, we can express $\mathrm{Li}_2(e^{\pi i/3}) + \mathrm{Li}_2(e^{-\pi i/3})$ as follows:

$$\begin{aligned}\mathrm{Li}_2(e^{\pi i/3}) + \mathrm{Li}_2(e^{-\pi i/3}) &= \frac{\pi^2}{108} - \frac{1}{18}\psi^{(1)}\left(\frac{1}{2}\right) + \frac{1}{36}\left\{\psi^{(1)}\left(\frac{1}{6}\right) + \psi^{(1)}\left(\frac{5}{6}\right)\right\} - \\ &\quad - \frac{1}{36}\left\{\psi^{(1)}\left(\frac{1}{3}\right) + \psi^{(1)}\left(\frac{2}{3}\right)\right\}, \end{aligned} \tag{D.31}$$

where $\psi^{(1)}(z)$ is defined by

$$\psi^{(1)}(z) = \frac{d^2}{dz^2}\ln\Gamma(z)\,. \tag{D.32}$$

Since

$$\psi^{(1)}(z) + \psi^{(1)}(1-z) = \frac{\pi^2}{\sin^2 \pi z}\,, \tag{D.33}$$

which follows from

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}\,, \tag{D.34}$$

we have

$$\mathrm{Li}_2(e^{\pi i/3}) + \mathrm{Li}_2(e^{-\pi i/3}) = \frac{\pi^2}{18}\,. \tag{D.35}$$

Therefore, the integral $I_{3/2}$ is given by

$$I_{3/2} = -\frac{\pi^2}{9}\,. \tag{D.36}$$

$\boldsymbol{V_{30} + V_{21} - V_{300} - V_{210} - V_{201}}$**.** Having finished all the preparations, we obtain the following expression for the sum of $V_{30/300}(n)$ and $V_{21/210+201}(n)$:

$$\begin{aligned}V_{30/300}(n) + V_{21/210+201}(n) &= \frac{2\pi\ln 2}{\epsilon} - 6\ln 2 + \frac{\pi^2}{3} - (\ln 2)^2 + \frac{(1-4n)\pi^2}{(2n)^2} + \\ &\quad + I_n + \left(\ln\frac{\sin\frac{\epsilon}{2n}}{\sin\frac{\epsilon}{n}}\right)^2 + o(\epsilon) \\ &= \frac{2\pi\ln 2}{\epsilon} - 6\ln 2 + \frac{\pi^2}{3} + \frac{(1-4n)\pi^2}{(2n)^2} + I_n + o(\epsilon)\,. \end{aligned} \tag{D.37}$$

From (D.9), (D.27), and (D.36), $V_{30} + V_{21} - V_{300} - V_{210} - V_{201}$ is given by

$$V_{30} + V_{21} - V_{300} - V_{210} - V_{201} = -\frac{7\pi^2}{36} + I_1 - I_{3/2} + o(\epsilon) = o(\epsilon)\,. \tag{D.38}$$

E. V_{111}

We calculate the integral

$$\int_{\pi/2}^{3\pi/2} d\theta_3 \int_{-\pi/2}^{\pi/2} d\theta_2 \int_{-3\pi/2}^{-\pi/2} d\theta_1 \left|3\sin\frac{\theta_2-\theta_1}{3}\right|^{-1} \left|3\sin\frac{\theta_3-\theta_1}{3}\right|^{-1} \left|3\sin\frac{\theta_3-\theta_2}{3}\right|^{-1}, \tag{E.1}$$

which is the value of V_{111} (4.46) in the limit $\epsilon \to 0$. The integral simplifies under the change of variables,

$$t_i = \sqrt{3}\tan\frac{\theta_i}{3} \qquad (i = 1, 2, 3)\,, \tag{E.2}$$

which corresponds to a conformal transformation to the upper-half plane:

$$\int_{\pi/2}^{3\pi/2} d\theta_3 \int_{-\pi/2}^{\pi/2} d\theta_2 \int_{-3\pi/2}^{-\pi/2} d\theta_1 \left|3\sin\frac{\theta_2-\theta_1}{3}\right|^{-1} \left|3\sin\frac{\theta_3-\theta_1}{3}\right|^{-1} \left|3\sin\frac{\theta_3-\theta_2}{3}\right|^{-1} = \\ = \int_1^{\infty} dt_3 \int_{-1}^{1} dt_2 \int_{-\infty}^{-1} dt_1 \, \frac{1}{(t_2-t_1)(t_3-t_2)(t_3-t_1)} \,. \tag{E.3}$$

The integrals with respect to t_1 and t_2 are easily carried out:

$$\begin{aligned} \int_1^{\infty} dt_3 \int_{-1}^{1} dt_2 \int_{-\infty}^{-1} dt_1 \, \frac{1}{(t_2-t_1)(t_3-t_2)(t_3-t_1)} &= \int_1^{\infty} dt_3 \int_{-1}^{1} dt_2 \, \frac{1}{(t_3-t_2)^2} \ln\frac{t_3+1}{t_2+1} \\ &= \int_1^{\infty} dt_3 \Big(\frac{1}{t_3-1} \ln\frac{t_3+1}{2} - \\ &\quad - \frac{1}{t_3+1} \ln\frac{t_3-1}{2} \Big). \end{aligned} \tag{E.4}$$

The integral over t_3 can be expressed in terms of the polylogarithm $\mathrm{Li}_2(z)$:[19]

$$\begin{aligned} \int_1^{\infty} dt_3 \left(\frac{1}{t_3-1} \ln\frac{t_3+1}{2} - \frac{1}{t_3+1} \ln\frac{t_3-1}{2} \right) &= \left[-2\,\mathrm{Li}_2\left(\frac{1-t_3}{2}\right) - \ln\frac{t_3-1}{2} \ln\frac{t_3+1}{2} \right]_{t_3=1}^{t_3=\infty} \\ &= \lim_{t_3\to\infty} \left\{ -2\,\mathrm{Li}_2\left(\frac{1-t_3}{2}\right) - \ln\frac{t_3-1}{2} \ln\frac{t_3+1}{2} \right\}. \end{aligned} \tag{E.5}$$

Using the formula[20]

$$\mathrm{Li}_2(-x) = -\frac{1}{2}\ln^2(x) - \frac{\pi^2}{6} + O\left(\frac{1}{x}\right) \qquad \text{for } x > 1 \,, \tag{E.6}$$

we can calculate the limit:

$$\lim_{t_3\to\infty} \left\{ -2\,\mathrm{Li}_2\left(\frac{1-t_3}{2}\right) - \ln\frac{t_3-1}{2} \ln\frac{t_3+1}{2} \right\} = \frac{\pi^2}{3} \,. \tag{E.7}$$

Therefore, we have

$$\int_{\pi/2}^{3\pi/2} d\theta_3 \int_{-\pi/2}^{\pi/2} d\theta_2 \int_{-3\pi/2}^{-\pi/2} d\theta_1 \left|3\sin\frac{\theta_2-\theta_1}{3}\right|^{-1} \left|3\sin\frac{\theta_3-\theta_1}{3}\right|^{-1} \left|3\sin\frac{\theta_3-\theta_2}{3}\right|^{-1} = \frac{\pi^2}{3} \,. \tag{E.8}$$

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